Part IA: Mathematics for Natural Sciences B
Examples Sheet 1: Lines, planes and the scalar product

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with a (†) require you to carefully review your lecture notes to make sure you understand basic definitions and properties.

Making friends

1. Speak to your supervision partner, and, if you don't know already, find out: (a) where they are from; (b) their favourite food; (c) what they like to do to relax.

Basics of vector algebra

2. Let $\overrightarrow{OA} = (2,3)$ and $\overrightarrow{AB} = (4,6)$. Compute $\overrightarrow{BO}$.
3. Suppose that an aeroplane's engine produces a velocity 125 km h^{-1} due North. How fast will the aeroplane travel across the surface of the Earth, and at what bearing from North, if there is a wind coming from the South-East at a speed of 80 km h^{-1}?
4. Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

The equation of a line

5. (†) Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be 3-vectors, and suppose that $\mathbf{w} \neq \mathbf{0}$.
   (a) Explain why the equation $\mathbf{r} = \mathbf{v} + \lambda \mathbf{w}$, as $\lambda \in \mathbb{R}$ varies, represents a line, and summarise its properties. Why is the condition $\mathbf{w} \neq \mathbf{0}$ necessary?
   (b) If $\mathbf{v} = (x_0, y_0, z_0)$ and $\mathbf{w} = (a, b, c)$, where $a, b, c \neq 0$, show that the same line may be equivalently described through the system of equations:
   $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
   What is the corresponding system of equations in the cases where one or more of a, b, c are zero?
   (c) Show that the position vectors $(1,0,1)$, $(1,1,0)$ and $(1,-3,4)$ lie on a straight line, and find both its vector form, as in (a), and its equational form, as in (b).
6. Show that the solution of the linear system $x + 2y + 3z = 0, 3x + 2y + z = 0$ is a line that is equally inclined to the x and z-axes, and makes an angle $\arccos(-\sqrt{2/3})$ with the y-axis.
7. A *median* of a triangle is a line joining a vertex to the midpoint of its opposite edge. Prove that the three medians of a triangle intersect.
8. Prove that in any tetrahedron, the lines joining the midpoints of opposite edges are concurrent.

Basic properties of the scalar product

9. (†) Let $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ be 3-vectors.
   (a) Give the definition of the *scalar product* $\mathbf{v} \cdot \mathbf{w}$ in terms of lengths and angles.
   (b) From this definition, prove each of the following properties of the scalar product:
      (i) *commutativity*: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;
      (ii) *homogeneity*: $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w})$;
      (iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.

10. (†) Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.
    (a) Show that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, for $i, j = 1, 2, 3$, where δ_{ij} is the *Kronecker delta*, defined by:
    $$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$
    (b) By writing $\mathbf{v}$, $\mathbf{w}$ as linear combinations of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and applying the properties of the scalar product from the previous question, prove the formula $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.
11. Compute the projection of the vector $\mathbf{a} = (1, 2, 3)$ in the direction $\mathbf{b} = (3, 2, 1)$.
12. Suppose that the points A, B have position vectors $\mathbf{a} = (0, 3, 4)$, $\mathbf{b} = (3, 2, 1)$ respectively. Determine the angles AOB and OAB.
13. Four points A, B, C, D are such that $AD \perp BC$ and $BD \perp AC$. Show that $CD \perp AB$.
14. (a) Prove the *Cauchy-Schwarz inequality* $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.
    (b) From the Cauchy-Schwarz inequality, deduce the *triangle inequality* $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. What is the geometrical significance of this inequality?
15. Prove that for any tetrahedron, the sum of the squares of the lengths of the edges equals four times the sum of the squares of the lengths of the lines joining the mid-points of opposite edges.

The equation of a plane

16. (†) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be fixed 3-vectors, with $\mathbf{b} \neq \mathbf{0}$. Explain why the equation $\mathbf{b} \cdot (\mathbf{r} - \mathbf{a}) = 0$ represents a plane, and summarise its properties. Show that:
    (a) the length of the projection of the vector $\mathbf{c}$ onto the plane is $\sqrt{|\mathbf{c}|^2 - (\hat{\mathbf{b}} \cdot \mathbf{c})^2}$;
    (b) the distance of the point represented by the position vector $\mathbf{c}$ to the plane is $|(\mathbf{c} - \mathbf{a}) \cdot \hat{\mathbf{b}}|$,

    where $\hat{\mathbf{b}}$ is a unit vector in the direction of $\mathbf{b}$.
17. Calculate the shortest distances between the plane $5x + 2y - 7z + 9 = 0$ and the points $(1, -1, 3)$ and $(3, 2, 3)$. Are the points on the same side of the plane?
18. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are of equal length l, and define the positions of the points A, B and C relative to O. If O, A, B and C are vertices of a regular tetrahedron, find the distance of any one vertex from the opposite face. *[Hint: consider the plane through A, B, C.]*
19. Find the acute angle at which two diagonals of a cube intersect. *[Hint: put the origin at the centre of the cube.]*

Equations of other 3D surfaces

20. (†) Let k, m, n be positive constants, with $m < 1$, and let $\mathbf{u}$ be a unit vector. Describe the geometry of the following surfaces: (a) $|\mathbf{r}| = k$; (b) $\mathbf{r} \cdot \mathbf{u} = m|\mathbf{r}|$; (c) $|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = n$.

Past paper questions

Normally, some past paper questions are included at the end of these examples sheets. However, in your first week, you don't know enough to attempt the past paper questions on vector geometry. Instead, this week:

- Look up the format of the first-year maths exams. How many questions do you have to answer?
- Think about timing for the exams. How long should you spend on each question?