Part IA: Mathematics for Natural Sciences B
Examples Sheet 2: The vector product, vector area and coordinate systems

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Questions marked with a (†) require you to carefully review your lecture notes to make sure you understand basic definitions and properties.

Basic properties of the vector product

1. (†) Let $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ be 3-vectors.
   (a) Give the geometrical definition of the *vector product* (or *cross product*) $\mathbf{v} \times \mathbf{w}$ in terms of lengths, angles and an appropriate perpendicular vector.
   (b) From this definition, prove each of the following properties of the vector product:
      (i) *anti-commutativity*: $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$;
      (ii) *homogeneity*: $(\lambda \mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w})$;
      (iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$.

2. (†) Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.
   (a) Show that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.
   (b) By writing $\mathbf{v}, \mathbf{w}$ as linear combinations of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and applying the properties of the vector product from the previous question, prove the formula:
   $$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, \quad v_3 w_1 - v_1 w_3, \quad v_1 w_2 - v_2 w_1).$$
   (c) Hence, write down a formula for computing $\mathbf{v} \times \mathbf{w}$ from the determinant of an appropriate 3×3 matrix.

3. Find the angle between the position vectors of the points $(2, 1, 1)$ and $(3, -1, -5)$, and find the direction cosines of a vector perpendicular to both. Can both the angle and vector be computed using *only* the vector product?

4. Describe the locus of the points that satisfy the equation $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (1, -1, 0)$.

5. Prove the identity $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$.

More on the equation of a plane

6. Find an equation of the form $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ for the plane passing through $(1, 1, 1)$, $(1, 2, 3)$ and $(0, 0, 4)$.

7. You need to drill a hole in a piece of metal starting at a right angle to a flat surface passing through the points $A = (1, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$, with the hole emerging at the point $D = (2, 1, 0)$. How long a drill must you use and where (in the plane ABC) must you start drilling?

Shortest distances

8. (†) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ be 3-vectors, with $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{c} \neq \mathbf{d}$. Derive, in terms of the vector product, formulae for the following:
   (a) the distance from the point $\mathbf{c}$ to the line passing through $\mathbf{a}, \mathbf{b}$;
   (b) the distance from the line passing through $\mathbf{a}, \mathbf{b}$ to the line passing through $\mathbf{c}, \mathbf{d}$.

9. Using the vector product, compute the shortest distance from a vertex of a cube to the diagonal excluding that vertex.

The vector triple product

10. (†) By expanding in terms of the standard basis vectors, prove *Lagrange's formula* for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Think of a way of remembering this formula off by heart - it is very useful! Hence construct an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

11. Prove the *Jacobi identity* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

12. Solve the vector equation $\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$ for $\mathbf{r}$, where $\lambda \neq 0$, and $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$ are arbitrary 3-vectors.

The scalar triple product

13. (†) Give the definition of the *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of the 3-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

    (a) Prove that the scalar triple product is symmetric on odd permutations of its entries, and is antisymmetric on even permutations of its entries. Deduce that $[\mathbf{a}, \mathbf{a}, \mathbf{c}] = [\mathbf{a}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{a}] = 0$.

    (b) Write down a formula for the scalar triple product as a determinant of an appropriate matrix.

    (c) Show that the volume of the parallelepiped with vertices $\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{c}, \mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. Hence explain why $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ implies that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are not coplanar, and thus form a basis.

14. Simplify the scalar triple products $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$

15. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be non-coplanar 3-vectors. We define the *reciprocal vectors* to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be the vectors:

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \qquad \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \qquad \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}.$$

    (a) Explain why $\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$, and $\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{c} = \mathbf{C} \cdot \mathbf{a} = \mathbf{C} \cdot \mathbf{b} = 0$.

    (b) Show that the vectors $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (0, 0, 1)$ and $\mathbf{c} = (2, -1, 1)$ form a non-orthogonal basis and, by appropriate use of the reciprocal vectors, write the vector $\mathbf{d} = (1, 1, 1)$ in terms of this basis.

Vector area

16. (†) Define the *vector area* $\mathbf{A}$ of a surface composed of k flat faces with areas $A_1, ..., A_k$ and unit normals $\hat{\mathbf{n}}_1, ..., \hat{\mathbf{n}}_k$. Give a very general explanation of how this could be extended to *curved surfaces*, and hence explain why we expect the vector area of any *closed* surface to be $\mathbf{0}$. What are the conventions usually used when choosing the unit normal(s)?

17. Compute the vector area of the square with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$, $(0, 2, 0)$, taken in that order. Hence compute the vector area of the pyramid extending this square with the point $(1, 1, 1)$, excluding its square face.

18. Compute the vector area of a lampshade (truncated hollow cone) bounded by a horizontal circle of radius 4 units and a horizontal circle of radius 3 units at some height above the first (note the result is independent of the height!).

19. (†) Let $\mathbf{S}$ be the vector area of the surface S.

    (a) Prove that the area of the projection of the surface S onto the plane with unit normal $\hat{\mathbf{m}}$ is $|\mathbf{S} \cdot \hat{\mathbf{m}}|$.

    (b) Compute the vector area of the projection of the square with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$, $(0, 2, 0)$ onto the plane with unit normal $\hat{\mathbf{m}} = (0, -1, 1)/\sqrt{2}$.

20. Compute the vector area of the loop with vertices $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 1, 1)$, $C = (0, 2, 0)$, taken in that order. *[Hint: consider filling the loop with three polygonal surfaces, parallel to the yz, xz, xy planes respectively.]* What is the area of the loop projected onto: (a) the plane with normal $(0, -1, 1)$; (b) the plane that maximises the projected area?

Coordinate systems

21. In 2D Cartesian coordinates, a circle is specified by $(x-1)^2 + y^2 = 1$. Find its equation in plane polar coordinates.
22. A point has Cartesian coordinates $(3, 4, 5)$. What are its cylindrical polar and spherical polar coordinates?
23. Let $a > 0$ be a constant. Describe the following loci:
    (a) (i) $\phi = a$; (ii) $r = \phi$, in plane polar coordinates.
    (b) (i) $z = a$; (ii) $r = a$; (iii) $r = a$ and $z = \phi$, in cylindrical polar coordinates.
    (c) (i) $\theta = a$; (ii) $\phi = a$; (iii) $r = a$; (iv) $r = \theta = a$, in spherical polar coordinates.

Past paper questions

Only attempt these questions before the supervision if you have lots of spare time. Sometimes, we might do them together in the supervision; otherwise, you can use these questions as revision material in the holidays.

Paper 2, Question 2, 2023 (2 marks)

The Cartesian coordinates of point A are $(x, y, z) = (-3, -4, -1)$. For this point A, find its (a) cylindrical polar coordinates, (r, θ, z), (b) spherical polar coordinates (r, θ, ϕ), where the angle θ is measured from the positive z-axis.

Paper 2, Question 4, 2023 (2 marks)

Find the vector area $\mathbf{S}_{ABCA}$ of the triangle ΔABC with vertices at $A(0, 0, 0)$, $B(1, 1, 0)$, and $C(1, 1, 1)$, travelling round the perimeter in the direction $A \to B \to C \to A$.

Paper 2, Question 11, 2022 (20 marks)

(a) Using the scalar triple product, give a condition for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ to form a basis.

Assuming that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ do form a basis, an arbitrary vector $\mathbf{y}$ can be written as $\mathbf{y} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$. Find the coefficients α, β and γ in terms of scalar triple products involving vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{y}$.

(b) A vector $\mathbf{x} \in \mathbb{R}^3$ satisfies the equation:

$$\mathbf{x} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{x})\mathbf{c}, \qquad (\dagger)$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-zero position vectors in $\mathbb{R}^3$.

(i) Take the cross product of (†) with the vector $\mathbf{c}$, and thus deduce an expression for $\mathbf{x}$ in terms of the vectors $\mathbf{a}$, $\mathbf{c}$ and a parameter λ.

(ii) To determine λ, substitute the expression for $\mathbf{x}$ you found in (b)(i) into (†) and thus find conditions on $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ for which λ has a unique value, has multiple values, or is undefined.

(iii) Solve (†) for $\mathbf{x}$ in the cases where solutions exist, and interpret these solutions geometrically.

Paper 2, Question 11, 2023 (20 marks)

(a) Let $\mathbf{a} = \hat{\mathbf{i}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{k}}$, where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are the Cartesian unit vector basis of the three-dimensional space. Show that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

(b) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three-dimensional vectors. Show that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ do not lie in the same plane if $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \neq 0$.

(c) Determine and justify your answer whether:

(i) the four points with position vectors $\mathbf{P}_1 = (0, 0, 2)$, $\mathbf{P}_2 = (0, 1, 3)$, $\mathbf{P}_3 = (1, 2, 3)$ and $\mathbf{P}_4 = (2, 3, 4)$ lie in the same plane,

(ii) the four points with position vectors $\mathbf{Q}_1 = (-2, 1, 1)$, $\mathbf{Q}_2 = (-1, 2, 2)$, $\mathbf{Q}_3 = (-3, 3, 2)$ and $\mathbf{Q}_4 = (-2, 4, 3)$ lie in the same plane.