Part IA: Mathematics for Natural Sciences B
Examples Sheet 3: Complex numbers

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with a (†) require you to carefully review your lecture notes to make sure you understand basic definitions and properties. Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions.

Real and imaginary parts

1. Find the real and imaginary parts of the following numbers (where n is an integer):

   (a) i^3, (b) i^{4n}, (c) $\left(\frac{1+i}{\sqrt{2}}\right)^2$, (d) $\left(\frac{1-i}{\sqrt{2}}\right)^2$, (e) $\left(\frac{1+\sqrt{3}i}{2}\right)^3$, (f) $\frac{1+i}{2-5i}$, (g) $\left(\frac{1+i}{1-i}\right)^2$.

2. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:

   (a) z^2, (b) iz, (c) $(1+i)z$, (d) $z^2(z-1)$.

3. Define u and v to be the real and imaginary parts, respectively, of the complex function $w = 1/z$. Show that the contours of constant u and v are circles. Show also that the contours of u and the contours of v intersect at right angles.

Factoring polynomials and solving equations

4. Factorise the following expressions: (a) $z^2 + 1$; (b) $z^2 - 2z + 2$; (c) $z^2 + i$; (d) $z^2 + (1-i)z - i$.

5. Consider the polynomial equation $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$, where the coefficients $a_n, a_{n-1}, ..., a_0$ are real. Show that the solutions to this equation come in complex conjugate pairs. Deduce that if n is odd, there is at least one real solution.

Geometry of complex numbers

6. (†) Using a diagram, explain the geometric meaning of the *modulus*, $|z|$, and *argument*, $\arg(z)$, of a complex number z. Find the moduli and (principal) arguments of: (a) $1 + \sqrt{3}i$; (b) $-1 + i$; (c) $-\sqrt{3} - i/\sqrt{3}$.

7. For $z \in \mathbb{C}$, show that $|z|^2 = zz^*$. Hence prove that $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$, where $a, b \in \mathbb{C}$, and interpret this result geometrically.

8. (†) By writing $z = |z|(\cos(\arg(z)) + i\sin(\arg(z)), w = |w|(\cos(\arg(w)) + i\sin(\arg(w))$, compute the modulus and argument of the product zw, and explain its geometrical significance.

9. Let $z_1 = 2 + i, z_2 = 3 + 4i$.

   (a) Plot the numbers $z_1, z_2, z_1^*, z_2^*, z_2 - z_1$ and $z_2 - 2z_1$ on an Argand diagram.

   (b) Find $z_1 z_2$ by: (i) adding arguments and multiplying moduli; (ii) using the rules of complex algebra.

10. Show that the expression $z_1 z_2^*$ is invariant under a rotation of z_1, z_2 about the origin.

Circles and lines

11. Describe the sets of points $z \in \mathbb{C}$ satisfying:

    (a) $|z| = 4$, (b) $|z-1| = 3$, (c) $|z-i| = 2$, (d) $|z-(1-2i)| = 3$, (e) $|z^*-1| = 1$, (f) $|z^*-i| = 1$.

12. Describe the set of points $z \in \mathbb{C}$ satisfying $|z-2-i| = 6$. Without further calculation, describe the sets of points $u \in \mathbb{C}, v \in \mathbb{C}$ satisfying $u = z + 5 - 8i$ and $v = iz + 2$, where $|z-2-i| = 6$.

13. Let $a, b \in \mathbb{C}$. Show that the set of points satisfying $|z-a| = \lambda|z-b|$, where $\lambda \neq 1$, is a circle in the complex plane. Determine the centre and radius of the circle $|z| = 2|z-2|$.

14. Describe the sets of points $z \in \mathbb{C}$ satisfying:

    (a) $|z-2| = |z+i|$, (b) $|z-2| = |z^* + i|$, (c) $\arg(z) = \pi/2$, (d) $\arg(z^*) = \pi/4$.

Exponential form of a complex number

15. (†) State *Euler's formula* for the complex exponential $e^{i\theta}$. Hence provide a simpler derivation of the modulus-argument multiplication law proved in Question 8.

16. Find (a) the real and imaginary parts; (b) the modulus and argument, of:

$$\frac{e^{i\omega t}}{R + i\omega L + (i\omega C)^{-1}},$$

where ω, t, R, L, C are real, quoting your answers in terms of $X = \omega L - (\omega C)^{-1}$.

17. Express each of the following in Cartesian form: (a) $e^{-i\pi/2}$; (b) $e^{-i\pi}$; (c) $e^{i\pi/4}$; (d) e^{1+i}; (e) $e^{2e^{i\pi/4}}$.

18. A simple harmonic oscillator has position $x(t) = 7 + 24\cos(3t) + 7\sin(3t)$ at time t.

    (a) Show that the displacement from the centre of motion (at $x = 7$) is given by the real part of $(24-7i)e^{3it}$. Hence compute the amplitude of the motion, and the distances of the stationary points of motion from the origin.

    (b) By writing the displacement from the centre of motion in the form $Ae^{i\phi}$, find the first two times at which the oscillator passes the origin after $t = 0$.

19. Explain why $e^{2\pi i n} = 1$ for all integers n. Hence find all natural logarithms of: (a) -1; (b) i; (c) $1+i$.

Roots of unity

20. (†) Write down the solutions to the equation $z^n = 1$ in terms of complex exponentials, and plot the solutions on an Argand diagram. *[Recall that the solutions are called the* nth roots of unity.*]*

21. Find all solutions of the following equations, and plot them on an Argand diagram: (a) $z^3 = -1$; (b) $z^4 = 1$; (c) $z^2 = i$; (d) $z^3 = -i$.

22. If $\omega^n = 1$, determine the possible values of $1 + \omega + \omega^2 + \cdots + \omega^{n-1}$, and interpret your result geometrically.

23. Show that the roots of the equation:

$$z^{2n} - 2bz^n + c = 0$$

will, for general complex values of b and c and integral values of n, lie on two circles in the Argand diagram. Give a condition on b and c such that the circles coincide. Find the largest possible value for $|z_1 - z_2|$, if z_1 and z_2 are roots of $z^6 - 2z^3 + 2 = 0$.

Trigonometry with complex numbers

24. (†) Prove *De Moivre's formula*, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$. Hence express $\sin(3\theta)$ in terms of powers of $\sin(\theta)$.

25. (†) Starting from Euler's formula, derive formulae for the trigonometric functions $\sin(\theta), \cos(\theta)$ in terms of complex exponentials. Hence express $\sin^5(\theta)$ in terms of $\sin(\theta), \sin(3\theta)$ and $\sin(5\theta)$.

26. Evaluate the following trigonometric sums: (a) $\displaystyle\sum_{n=1}^{5} \sin(n\theta)$; (b) $\displaystyle\sum_{n=1}^{5} r^n \cos(n\theta)$.

Past paper questions

Only attempt these questions before the supervision if you have lots of spare time. Sometimes, we might do them together in the supervision; otherwise, you can use these questions as revision material in the holidays.

Paper 1, Question 11, 2022 (20 marks)

(a) Find all possible real and imaginary parts of the following expressions: (i) $(i^i)^i$; (ii) $i^{(i^i)}$.

(b) Describe, with the aid of a sketch, the curve in the Argand diagram whose equation is $|z + 1 + i| = 8$.

(c) Describe, with the aid of sketches, the loci determined, for z on the curve in part (b), by the complex numbers:

(i) $u = \frac{3}{2}z + \frac{1}{2}z^*$;

(ii) $v = u + 4 + 3i$;

(iii) $w = iv$.

(d) Express $\sin(5\theta)$ in terms of $\sin(\theta)$ and its powers, and find the values of θ such that $16\sin^5(\theta) = \sin(5\theta)$ for $0 \leq \theta < 2\pi$.

Paper 1, Question 11, 2021 (20 marks)

(a) Find the real and imaginary parts of the following complex numbers: (i) $(1/2 + 2i)^2$; (ii) $(1/2 + 2i)^4$.

In the remainder of the question, a point $z = t + i/t$ moves in the complex plane as the real parameter t increases continuously from $t = 1/2$ to $t = 2$.

(b) In the following three cases, sketch the trajectories of the given points as t varies, showing the direction of their motion by arrows, and giving the coordinates of the initial and final positions, and also of points (if any) where the trajectories cross the axes:

(i) z and z^*; where z^* is the complex conjugate of z;

(ii) $u = z^2$;

(iii) $v = z^4$.

(c) For $v = z^4$, find a relation between the real and imaginary parts of v, and thus demonstrate that $\mathrm{Re}(v)$ is a quadratic function of $\mathrm{Im}(v)$.

Paper 1, Question 11, 2019 (20 marks)

(a) Use De Moivre's theorem to express $8\cos(6\theta) + 15\sin(4\theta)\sin(2\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

(b) For $z \in \mathbb{C}$, with $z = x + iy$, find the real and imaginary parts of $\tan(z^*)$.

(c) Find the locus that solves $|2z - z^* - 3i| = 2$, and sketch it on an Argand diagram.

(d) By writing $z = re^{i\theta}$ (where $r, \theta \in \mathbb{R}$), find the real and imaginary parts of $f = \log(z^{1+i})$ in terms of r and θ. By limiting the argument of z to range from $-\pi$ to π, sketch on an Argand diagram the solution to $\mathrm{Re}(f) = 0$.