# Part IA: Mathematics for Natural Sciences B
# Examples Sheet 4: Hyperbolic functions and differential calculus

*Please send all comments and corrections to jmm232@cam.ac.uk.*

Questions marked with a (†) require you to carefully review your lecture notes to make sure you understand basic definitions and properties. Questions marked with a (∗) are difficult and should not be attempted at the expense of the other questions.

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### Hyperbolic functions

1. (†) Give the definitions of $\sinh(x)$ and $\cosh(x)$ in terms of exponentials. Hence show that:

   (a) $\cos(x) = \cosh(ix)$, (b) $i\sin(x) = \sinh(ix)$.

   Deduce *Osborn's rule*: 'a hyperbolic trigonometric identity can be deduced from a circular trigonometric identity by replacing each trigonometric function with its hyperbolic counterpart *except* in places where sine enters quadratically, where we must include an additional factor of $-1$.'

   Hence, using Osborn's rule, write down the formula for $\tanh(x+y)$ in terms of $\tanh(x), \tanh(y)$.

2. Let $b \geq a > 0$ be fixed, and let $\theta$ be a variable parameter. Find the Cartesian equations of the two parametric curves: (a) $(x, y) = (a\cos(\theta), b\sin(\theta))$; (b) $(x, y) = (a\cosh(\theta), b\sinh(\theta))$, and sketch them in the plane.

3. (†) Express $\cosh^{-1}(x), \sinh^{-1}(x)$ and $\tanh^{-1}(x)$ as logarithms.

### Basic definitions and notation for derivatives

4. (†) Let $y \equiv y(x)$ be a function of $x$. Define the *derivative* $dy/dx$ of $y$ as a limit.

   (a) Using the limit definition, show that differentiation is a *linear* operation, that is, for any constants $c_1, c_2$ and any functions $y_1 \equiv y_1(x), y_2 \equiv y_2(x)$, we have:

   $$\frac{d}{dx}(c_1 y_1 + c_2 y_2) = c_1 \frac{dy_1}{dx} + c_2 \frac{dy_2}{dx}.$$

   (b) Using the limit definition, find the derivative of $y(x) = x^n$, for $n = 0, 1, 2, 3, ...$.

5. Find the derivatives of the following functions, where $a, b, c, \theta$ are real constants and $n = 0, 1, 2, 3, ...$:

   (a) $x^2 + 3$, (b) $x^4 + 2x + 6$, (c) $ax^n + bx + c$, (d) $ax + bx^2\sin(\theta)$.

6. Let $y \equiv y(t)$ be a function of $t$, determined through the differential equation:

   $$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 0.$$

   Rewrite this differential equation using (a) Newton's notation for derivatives; (b) Lagrange's notation for derivatives.

7. Using the limit definition of differentiation, show that for $a > 0$, the derivative of $y(x) = a^x$ is proportional to $a^x$. *[Recall that by definition, $e$ is the value for which the proportionality constant is one.[1]]*

[1] Proving that such a number exists and is unique is very hard, though, so we'll just assume this.

## Rules of differentiation

8. (†) Using the limit definition of the derivative, prove the following rules of differentiation:

   (a) THE CHAIN RULE.
   $$(f(g(x)))' = g'(x)f'(g(x)).$$

   (b) THE PRODUCT RULE.
   $$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

   (c) THE QUOTIENT RULE.
   $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}.$$

   (d) THE INVERSE FUNCTION RULE.
   $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

   *[Hint: For (c), apply the product and chain rules to the product $f(x) \cdot g(x)^{-1}$. For (d), apply the chain rule to $f^{-1}(f(x)) = x$.]*

   ($*$) Explain how the inverse function rule is related to the *reciprocal rule* in Leibniz's notation for derivatives:
   $$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}.$$

9. (†) Using the inverse function rule, and the derivative of $e^x$, prove that the derivative of $\log(x)$ is $1/x$.

10. Compute the derivatives of the following compound functions:

    (a) $3^{x^2}$, (b) $\dfrac{e^x}{x^3 - 1}$, (c) $x^3 \log(x^2 - 7)$, (d) $\sqrt{x^3 - e^x \log(x)}$.

11. (†) By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives:

    (a) $\cos(x)$, (b) $\sin(x)$, (c) $\cosh(x)$, (d) $\sinh(x)$, (e) $\tan(x)$, (f) $\tanh(x)$.

12. (†) Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived in Question 3; (ii) the inverse function rule, compute the derivatives of:

    (a) $\cosh^{-1}(x)$, (b) $\sinh^{-1}(x)$, (c) $\tanh^{-1}(x)$.

13. Using: (a) implicit differentiation; (b) the reciprocal rule, find $dy/dx$ given $y + e^y \sin(y) = 1/x$, and make sure that your answers agree.

## Curve-sketching

14. (†) State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:

    (a) $x$, (b) $\sin(x)$, (c) $e^x$, (d) $\sin(\frac{\pi}{2} - x)$, (e) $|x|\cos(x)$, (f) $\sqrt{x}$, (g) $2$, (h) $0$, (i) $\log\left|\dfrac{1+x}{1-x}\right|$.

15. (†) Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!

16. (†) Sketch the graphs of the following hyperbolic functions, explaining your reasoning in each case:

    (a) $\cosh(x)$, (b) $\sinh(x)$, (c) $\tanh(x)$, (d) $\cosh^{-1}(x)$, (e) $\sinh^{-1}(x)$, (f) $\tanh^{-1}(x)$.

17. Sketch the graphs of the following functions, explaining your reasoning in each case:

    (a) $(x-3)^3+2x$, (b) $\dfrac{x}{1+x^2}$, (c) $xe^x$, (d) $\dfrac{\log(x)}{1+x}$, (e) $\dfrac{1}{1-e^x}$, (f) $e^x\cos(x)$.

**Leibniz's formula**

18. (†) Using mathematical induction, prove *Leibniz's formula* for the $n$th derivative of a product:

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

where $f^{(k)}$ denotes the $k$th derivative of $f$. Hence compute the third derivative of $\log^2(x)$.

19. Use Leibniz's formula to prove that:

$$Z_n = \frac{d^n}{dx^n} e^{-x^2/2}$$

is a solution of the differential equation:

$$\frac{d^2 Z_n}{dx^2} + x\frac{dZ_n}{dx} + (n+1)Z_n = 0.$$

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**Past paper questions**

Only attempt these questions before the supervision if you have lots of spare time. Sometimes, we might do them together in the supervision; otherwise, you can use these questions as revision material in the holidays.

*Paper 1, Question 7, 2023 (2 marks)*
Find the stationary values of the function $f(x) = \exp(x/(1+x^2))$ and the values of $x$ at which they occur.

*Paper 1, Question 8, 2023 (2 marks)*
Given that $x + y + e^x + e^y = c$ where $c$ is a constant, find $dy/dx$ in terms of $x$ and $y$.

*Paper 1, Question 6, 2022 (2 marks)*
Find the value of $x$ at which the function $y = x^3 e^{-x}$ reaches its maximum in the range $0 \leq x < \infty$ and evaluate the value of $y$ at this point.

*Paper 1, Question 7, 2022 (2 marks)*
Sketch $y = \sin(x)/x^2$ for positive $x$ and label the crossing points, if any, with the horizontal axis.

*Paper 1, Question 11, 2023 (20 marks)*

(a) Find the real and imaginary parts of (i) $(3-2i)(4+3i)$, (ii) $(3-2i)/(4+3i)$.

(b) Find and sketch the set of points in the Argand diagram that satisfy the following equations with $z = x + iy$: (i) $\mathrm{Re}(z^2) = \mathrm{Im}(z^2)$, (ii) $\mathrm{Im}(z^2)/z^2 = -i$.

(c) Solve, for real $x$, the equation $\cosh(x) = \sinh(x) + 2\mathrm{sech}(x)$.

(d) Find all distinct solutions for $z$ of the equation $\cosh(z) = i$.