Part IA: Mathematics for Natural Sciences B
Examples Sheet 7: Integral calculus, multiple integrals and Gaussian integrals

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with a (†) require you to carefully review your lecture notes to make sure you understand basic definitions and properties. Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions.

Basic integrals

1. (†) *Write down* the indefinite integrals of each of the following functions, where $a \neq 0, n \neq -1$ is an integer, and f is any (differentiable, non-zero) function:

   (a) $(ax+b)^n$, (b) e^{ax+b}, (c) $(ax+b)^{-1}$, (d) $\sin(ax+b)$, (e) $\cos(ax+b)$,

   (f) $\sec^2(ax+b)$, (g) $\sinh(ax+b)$, (h) $\cosh(ax+b)$, (i) $f'(x)/f(x)$.

   Learn these integrals off by heart, and get your supervision partner to test you on them.

2. Using the results of the previous question, evaluate the definite integrals:

   (a) $\displaystyle\int_0^2 (x-1)^2\,dx$, (b) $\displaystyle\int_0^\pi e^{i\theta}\,d\theta$, (c) $\displaystyle\int_0^\pi \cos(x)\,dx$, (d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$, (e) $\displaystyle\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx$.

3. By writing both $\sin(x)$ and $\sinh(x)$ as exponentials, find the indefinite integral of $\sin(x)\sinh(x)$.

Integration by substitution

4. (†) By means of an appropriate trigonometric or hyperbolic substitution in each case, determine the indefinite integrals of the following functions:

   (a) $\dfrac{1}{\sqrt{1-x^2}}$, (b) $\dfrac{1}{\sqrt{x^2-1}}$, (c) $\dfrac{1}{\sqrt{1+x^2}}$, (d) $\dfrac{1}{1+x^2}$, (e) $\dfrac{1}{1-x^2}$

   Learn these integrals off by heart, and get your supervision partner to test you on them.

5. Using the results of the previous question, determine: (a) $\displaystyle\int \frac{1}{\sqrt{x^2+x+1}}$; (b) $\displaystyle\int \frac{8-2x}{\sqrt{6x-x^2}}$.

6. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

   (a) $x\sqrt{x+3}$, (b) $\tan(x)\sqrt{\sec(x)}$, (c) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$, (d) $\dfrac{1}{x\sqrt{x^2-1}}$.

7. This question shows that any trigonometric integral can be turned into an algebraic integral through the use of the powerful *half-tangent substitution*.

   (a) Show that if $t = \tan\left(\frac{1}{2}x\right)$, then $\sin(x) = 2t/(1+t^2)$, $\cos(x) = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$. Deduce that for any function f, we have:

   $$\int f\left(\sin(x),\cos(x)\right)\,dx = \int f\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

   (b) Using the method derived in (a), find the indefinite integrals of the following functions:

   (i) $\sec(x)$, (ii) $\operatorname{cosec}(x)$, (iii) $\dfrac{1}{2+\cos(x)}$.

Partial fractions and rational functions

8. (†) Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

$$\text{(a) } \frac{1}{1-x^2}, \qquad \text{(b) } \frac{3x}{2x^2+x-1}, \qquad \text{(c) } \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}.$$

Compare your answer to (a) with your answer to Question 4(e), where you evaluated the same integral using a substitution. Are your results compatible?

Integration by parts

9. Using integration by parts, determine the following integrals:

$$\text{(a) } \int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx, \qquad \text{(b) } \int_0^\infty xe^{-2x}\,dx, \qquad \text{(c) } \int_0^1 x\log\left(\frac{1}{x}\right)\,dx, \qquad \text{(d) } \int_0^\infty x^3e^{-x^2}\,dx.$$

10. By writing each of the following functions $f(x)$ in the form $1\cdot f(x)$, and using integration by parts, determine their indefinite integrals:

$$\text{(a) } \log(x), \qquad \text{(b) } \log^3(x), \qquad \text{(c) } \cosh^{-1}(x), \qquad \text{(d) } \tanh^{-1}(x), \qquad \text{(e) } \sin(\log(x)).$$

11. Let $I = \int_0^x e^{at}\cos(bt)\,dt$ and $J = \int_0^x e^{at}\sin(bt)\,dt$, where a and b are real constants, with $b \neq 0$.

    (a) Using integration by parts, show that $bI = e^{ax}\sin(bx) - aJ$. Find another relationship between I and J, and hence find I and J separately.

    (b) Now instead, evaluate the complex integral $I + iJ$ directly, and hence find I and J separately.

Leibniz's integral rule

12. (†) Using the multivariable chain rule, derive *Leibniz's integral rule*:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)\,dt.$$

Verify that the rule holds in the case where $a(x)=0$, $b(x)=1+x$, and $f(x,t)=t(x-t)$.

13. This question determines the Gaussian integral in a different way to lectures. Define:

$$f(x) = \left(\int_0^x e^{-t^2}\right)^2, \qquad \text{and} \qquad g(x) = \int_0^\infty \frac{e^{-x^2(t^2+1)}}{1+t^2}\,dt.$$

Use Leibniz's integral rule to show that $f'(x)+g'(x)=0$. Deduce that $f(x)+g(x)=\pi/4$, and conclude that:

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

Integral inequalities

14. Using a sketch, explain why $\sin(x) \geq 2x/\pi$ in the interval $0 \leq x \leq \pi/2$. Hence show that:

$$\int_0^{\pi/2} \frac{x^2}{1+\sin^2(x)}\, dx < \frac{\pi^3}{8}\left(1 - \frac{\pi}{4}\right).$$

15. (†) Clearly state *Schwarz's inequality* for integrals, and use it to show that:

$$\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{x^2+1}}\, dx < \frac{\pi}{4} \arctan\left(\frac{\pi}{2}\right).$$

Derive an alternative bound using $\sin(x) \leq x$ for $x \geq 0$, and comment on which approach yields the tightest bound.

Multiple integrals

16. (†) Explain the steps you would take to perform the integral of a function $f : D \to \mathbb{R}$ over an area D of the plane using Cartesian coordinates. Explain also, with the help of a diagram, how you would change the order of the double integration.

Hence evaluate:

$$\iint_D 2xy^2\, dA$$

over the triangle D bounded by the lines $y = 1$, $y = \frac{1}{2}x$ and $x = 0$ by (a) integrating with respect to y first, then with respect to x; (b) integrating with respect to x first, then with respect to y.

17. Evaluate the following integrals, including a suitable sketch of the regions in each case:

(a) $\iint_{D_1} \sin(x+y)\, dA$, (b) $\iint_{D_2} xe^{xy}\, dA$, (c) $\iint_{D_3} \sin(x+y)\, dA$, (d) $\iint_{D_4} x^2 y\, dA$.

where:

- $D_1 = [0, \frac{1}{2}\pi] \times [0, \frac{1}{2}\pi]$,
- $D_2 = [0, \pi] \times [0, \pi]$,
- D_3 is the triangle bounded by the lines $y = 0, x = \frac{1}{2}\pi, y = x$,
- D_4 is the region contained in the first quadrant of the plane, bounded by the curves $y = 2, y = x^2$ and $y = 1/x$.

18. (†) With the aid of a diagram, explain the formula for the *area element* dA in plane polar coordinates. Similarly, explain the formula for the *volume element* dV in cylindrical polar coordinates and in spherical polar coordinates. Using appropriate coordinates in each case, determine using multiple integration:

    (a) the area of a disk of radius a;

    (b) the volume of a flat-topped cone with height a, and opening angle $\pi/4$;

    (c) the volume of a round-topped cone, formed by the cone in (b) whose base is replaced by a sector of a sphere of radius a.

19. Evaluate the following integrals, including a suitable sketch of the regions in each case:

$$\text{(a)} \iint\limits_{D_1} x^2(1 - x^2 - y^2)\, dA, \qquad \text{(b)} \iint\limits_{D_2} \sqrt{x^2 + y^2}\, dA.$$

where:

- D_1 is the unit disk,
- D_2 is the region bounded by the curve $r = 2a(1 + \cos(\phi))$ in plane polar coordinates, with a constant.

20. (a) Find the volume of the cylindrically symmetric object bounded by the planes $z = 0$, $z = H$ and the surface of revolution $r = Re^{-z/H}$, where R, H are constants.

    (b) Now suppose that the cross-section of the shape is altered so that the cylindrical symmetry is broken. Specifically, suppose that a sector of each cross-sectional disk is removed such that at $z = H$, half the disk is removed, but at $z = 0$, none of the disk is removed, and in-between the amount of area removed interpolates linearly between these two extremes. Calculate the volume of the resulting object.

21. (†) Explain how to compute the mass of an object which occupies a volume V and has variable density $\rho \equiv \rho(x, y, z)$. Hence find the mass of a spherical body of radius R whose density is:

$$\rho(r) = Ar^{-2}\left(1 - e^{-r/r_0}\right),$$

where r is radial distance from the centre of the object, and A, r_0 are positive constants.

22. (∗) Show that:

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dxdy \neq \int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dydx.$$

Why can this happen here?

Gaussian integrals

23. (†) By considering the double integral of $e^{-(x^2+y^2)/a^2}$ over all of $\mathbb{R}^2$, prove that:

$$\int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx = a\sqrt{\pi}.$$

Hence evaluate:

$$\iiint\limits_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)/a^2}\, dV.$$

24. Prove by induction that for positive n and positive a:

$$\int_0^{\infty} x^{2n} e^{-ax^2}\, dx = \frac{(2n - 1)(2n - 3)...1}{2(2a)^n}\sqrt{\frac{\pi}{a}}.$$

Verify this result by repeatedly differentiating $\int_0^{\infty} e^{-ax^2}\, dx$ with respect to a.

Past paper questions

Only attempt these questions before the supervision if you have lots of spare time. Sometimes, we might do them together in the supervision; otherwise, you can use these questions as revision material in the holidays.

Paper 1, Question 9, 2023 (2 marks)
Show that:

$$\int_2^3 \frac{2x+1}{x(x+1)}\,dx = \log(n),$$

for some integer n that you should determine.

Paper 1, Question 8, 2022 (2 marks)
Find the indefinite integral of $y = e^{\sqrt{x}}/2\sqrt{x}$ for $x > 0$.

Paper 2, Question 9, 2022 (2 marks)
Find the indefinite integral of the function $x\log(x)$ for $x > 0$.

Paper 1, Question 8, 2021 (2 marks)
Find the indefinite integral of $x^3 e^{-x^4}$.

Paper 2, Question 18, 2023 (20 marks)

(a) Show that for integers $n > 1$,

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx + C_1,$$

$$\int \cos^n(x)\,dx = +\frac{1}{n}\sin(x)\cos^{n-1}(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx + C_2,$$

where C_1 and C_2 are arbitrary constants. Hence find $\int \sin^6(x)\,dx$ and $\int \cos^6(x)\,dx$.

(b) Using the above, show that:

$$\int_0^{\pi/2} \sin^n(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx, \qquad \int_0^{\pi/2} \cos^n(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2}(x)\,dx.$$

Hence evaluate $\int_0^{\pi/2} \sin^2(x)\,dx$, $\int_0^{\pi/2} \sin^4(x)\,dx$, $\int_0^{\pi/2} \cos^2(x)\,dx$ and $\int_0^{\pi/2} \cos^4(x)\,dx$.

Paper 1, Question 20(a), 2022 (6 marks)
For general functions $f(x)$, $g(x)$ and $h(x,t)$, write down the formula for:

$$\frac{d}{dx}\int_{g(x)}^{f(x)} h(x,t)\,dt,$$

and hence evaluate the limit:

$$\lim_{x\to\infty}\frac{d}{dx}\int_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}\,dt.$$

Paper 1, Question 12, 2022 (20 marks)

(a) Consider $\int_{x=2y}^{2} \int_{y=0}^{1} x^2 y^2 dx dy.$

(i) Explain why the integration should be done over x first.

(ii) Sketch the region of integration.

(iii) Evaluate the integral.

(iv) Change the order of integration and re-calculate the integral.

(b) The radial ($r \geq 0$) cross-section of a cup is shown in the diagram.

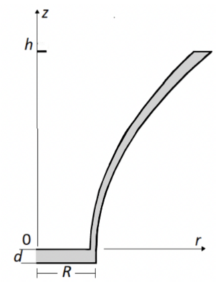

Figure 1: *Reproduced from University of Cambridge Natural Sciences Tripos, Part IA Mathematics Paper 1, Question 12, 2022.*

You may assume cylindrical symmetry about the z-axis. The outer curved line ($z \geq 0$) obeys $r = R(1 + z^2/h^2)$. The inner curved line obeys $r = 0.9R(1 + z^2/h^2)$. The cup is made from metal of density ρ. Calculate the mass of the cup.

Paper 1, Question 12, 2019 (20 marks)

A new rail link is planned. To support the tracks, the link requires the construction of an embankment across a small valley.

(a) The material for the embankment is stockpiled in a single circular heap of volume V and constant outer radius R. Within the heap, the height profile is given by $R\exp(-r/R)$ with $r \leq R$. Here, r is the radial distance from the vertical axis of the heap. Determine R in terms of V.

(b) The embankment is defined by the region $|x| \leq L$, $|y| \leq b(2 - z/H)$ and $H(x/L)^2 \leq z \leq H$, where x is oriented across the valley and z upwards. Here, H is the height of the embankment, $2L$ its length, and $2b$ its width at the top. Calculate the volume of the embankment.

(c) Later, it is realised that there needs to be a tunnel through the embankment for a cycle path along the bottom of the valley. The tunnel is cylindrical with diameter D (where $D < L^2/H$ and $D < H$) and is centred on $x = 0$, $z = \frac{1}{2}D$ and has its axis in the y-direction. Looking along the y-axis, sketch in the xz plane the embankment and the tunnel. Calculate the volume of the material that needs to be removed from the embankment to create the tunnel.