Part IA: Mathematics for Natural Sciences A
Examples Sheet 11: Linear ordinary differential equations

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Basic definitions

1. Consider the general linear nth-order ordinary differential equation:

$$\alpha_n(x)\frac{d^ny}{dx^n} + \alpha_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \alpha_1(x)\frac{dy}{dx} + \alpha_0(x)y = f(x).$$

   where $\alpha_n(x) \not\equiv 0$.

   (a) Give the definitions of the following terms: (i) homogeneous equation; (ii) coefficient functions; (iii) forcing.

   (b) Define a *complementary function* for this equation. How many arbitrary constants feature in the complementary function for this equation?

   (c) Define a *particular integral* for this equation. Is a particular integral for this equation unique?

   (d) Show that if y_{CF} is the complementary function for this equation, and y_{PI} is a particular integral, then the sum $y = y_{\mathsf{CF}} + y_{\mathsf{PI}}$ solves the equation.

   (e) Suppose that we now seek a particular solution of this equation satisfying certain boundary conditions. How many boundary conditions are needed to fully specify a particular solution?

2. By direct differentiation, verify that the following ordinary differential equations have the given complementary functions:

   (a) $y_{\mathsf{CF}} = Ax + Be^x$ is the complementary function for $(x-1)y'' - xy' + y = 0$;

   (b) $y_{\mathsf{CF}} = A + B\log(x)$ is the complementary function for $xy'' + y' = \cos(x)e^{x^2}$;

   (c) $y_{\mathsf{CF}} = Ax + B\sin(x)$ is the complementary function for $(1 - x\cot(x))y'' - xy' + y = x$;

   (d) $y_{\mathsf{CF}} = A + Bx + Ce^x$ is the complementary function for $y''' - y'' = x$.

3. By direct differentiation, verify that the following ordinary differential equations have the given particular integrals:

   (a) $y_{\mathsf{PI}} = \cos(x)$ is a particular integral for $-y'' + y = 2\cos(x)$;

   (b) $y_{\mathsf{PI}} = x^2$ is a particular integral for $xy'' + y' = 4x$;

   (c) $y_{\mathsf{PI}} = e^{x^2}$ is a particular integral for $y''' - 2xy'' - 2y' - y = (4x-1)e^{x^2}$;

   (d) $y_{\mathsf{PI}} = \sin(x)/x$ is a particular integral for $xy^{(4)} + 4y^{(3)} + xy^{(2)} + 2y^{(1)} + xy = \sin(x)$.

4. Verify that the equation:

$$(3+x)y'' + (2+x)y' - y = x^2 + 6x + 6$$

   has complementary function $y_{\mathsf{CF}}(x) = Ae^{-x} + B(x+2)$. Hence, by finding a particular integral of the form

$$y_{\mathsf{PI}}(x) = \alpha x^2 + \beta x + \gamma,$$

   determine the full solution to the equation subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$.

Constant coefficient equations

5. Consider the linear second-order ordinary differential equation with *constant coefficients*:

$$\alpha\frac{d^2y}{dx^2}+\beta\frac{dy}{dx}+\gamma y=f(x),$$

where α,β,γ are *constants*, with $\alpha\neq 0$.

   (a) Show that $y=e^{\mu x}$ solves the homogeneous equation if and only if μ satisfies the *auxiliary equation*:

$$\mu^2+\beta\mu+\gamma=0.$$

   Deduce the complementary function of the equation in the case that the auxiliary equation does not have repeated roots.

   (b) Show that if the auxiliary equation has a repeated root ω, then the complementary function is given by

$$y_{\mathsf{CF}}=(Ax+B)e^{\omega x}.$$

   (c) How do these results generalise to nth order linear differential equations with constant coefficients?

6. Determine the solutions of the following differential equations:

   (a) $y''+6y'+5y=0$;
   (b) $y''+3y'+4y=0$;
   (c) $y''+4y=x$;
   (d) $y''-2y'+2y=2x^2$;
   (e) $y''+y=|x|$;
   (f) $y''+3y'+2y=e^{-x}$;
   (g) $y''-2y'+5y=e^x\cos(2x)$;
   (h) $y''+2y'+y=2xe^{-x}$.

7. Determine the solutions of the following differential equations subject to the given constraints:

   (a) $y''-4y'+13y=0$, subject to $y(0)=\pi$ and $y(-\pi/2)=1$;
   (b) $y''-4y'+5y=125x^2$, subject to $y(0)=1$ and $y\left(\frac{\pi}{2}\right)=\frac{25\pi^2}{4}+20\pi+22$;
   (c) $y''+7y'+12y=6$, subject to $y(0)=0$ and $y\left(\frac{1}{3}\right)=\frac{1-e^{-1}}{2}$;
   (d) $y''+7y'+12y=2e^{-3x}$, subject to $y(0)=1$ and $y'(0)=0$.

8. Find the value of a for which the complementary function of the ODE:

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+ay=0,$$

is given by $y_{\mathsf{CF}}=Axe^{-2x}+Be^{-2x}$.

9. Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2}+y=\cos(kx),$$

where k is a real number.

10. The differential operator $\mathcal{L}$ is defined by:

$$\mathcal{L} = \frac{d^2}{dx^2} + \sqrt{3}\frac{d}{dx} + 3.$$

Solve the equation $\mathcal{L}y = 0$, and hence solve the equations:

(a) $\mathcal{L}y = e^{-\sqrt{3}x}$;

(b) $\mathcal{L}y = x$.

Without further calculation, state the general solution of $\mathcal{L}y = 2x + e^{-\sqrt{3}x}$. Find also the solution of this equation satisfying the boundary conditions $y(0) = 0, y(\pi) = \frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}}$.

Harmonic oscillators

11. Consider the constant coefficient linear second-order ordinary differential equation:

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + \omega_0^2 y = f(t),$$

modelling an oscillating system which depends on time t. The coefficients γ, ω_0 are positive.

(a) What is the physical interpretation of the constant γ? What is the physical interpretation of the function $f(t)$?

(b) Find the complementary function of this equation. Discuss the different forms the complementary function can take (in particular, defining the terms *underdamping*, *critical damping*, and *overdamping*), and how this relates to the *transient* behaviour of the oscillator.

(c) In the underdamped case, find the long-term behaviour of the oscillator in the case of resonant forcing:

$$f(t) = e^{-\gamma t}\sin\left(t\sqrt{\omega_0^2 - \gamma^2}\right).$$

Coupled systems of differential equations

[*This section is labelled 'non-examinable' in the lecture notes, but has appeared on Tripos papers - see e.g. 2023 Paper 2 or 2021 Paper 2.*]

12. (a) Consider the system of differential equations:

$$\frac{dx}{dt} = ax + by + p, \qquad \frac{dy}{dt} = cx + dy + q,$$

for the variables $x(t), y(t)$, where a, b, c, d, p, q are constants. Show that:

$$\frac{d^2x}{dt^2} = (a+d)\frac{dx}{dt} + (bc - ad)x + bq - pd.$$

(b) Hence:

(i) Find the general solution of the system:

$$\frac{dx}{dt} = 4y + 2, \qquad \frac{dy}{dt} = x.$$

(ii) Solve the system:

$$\frac{dx}{dt} = 3x - y, \qquad \frac{dy}{dt} = x + y,$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 1$.

(iii) Solve the system:

$$\frac{dx}{dt} = -3x + y, \qquad \frac{dy}{dt} = -5x + y,$$

subject to the initial conditions $x(0) = 1, y(0) = 1$.