Part IA: Mathematics for Natural Sciences A
Examples Sheet 12: Partial differentiation, and the chain rule for multivariable functions

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Partial differentiation: basic examples and properties

1. Let $f \equiv f(x,y)$ be a function of x and y.
   (a) Determine the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of the following functions.
   (i) $f = x^3 - 2x^2y + 3xy^3 - 4y^3$, (ii) $f = \exp(-x^2y^2)$, (iii) $f = \exp(-x/y)$, (iv) $f = \sin(x+y)$.
   (b) For each of the functions in part (a), compute the four possible second partial derivatives. Verify that in each case we have symmetry of the mixed partial derivatives.

2. Show that the function:
$$w(x,y) = \frac{1}{360}\left(15x^4y^2 - x^6 + 15x^2y^4 - y^6\right)$$
is a solution of the equation:
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = x^2y^2.$$

3. (**Reciprocity and the cyclic relation**) Three variables x, y, z are related by the implicit equation $f(x,y,z) = 0$ where f is some multivariable function.
   (a) Derive the *reciprocity relation* and the *cyclic relation* for the partial derivatives:
$$\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial y}\right)_z = 1, \qquad \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x = -1.$$
   (b) Verify that these relationships hold if: (i) $f(x,y,z) = xyz + x^3 + y^4 + z^5$; (ii) $f(x,y,z) = xyz - \sinh(x+z)$.

The single-variable chain rule, with multivariable functions

[*This section is not really lectured, but comes up a lot in exams! It's also a good thing to try before getting to the - more complicated - multivariable chain rule. You **don't** need to have studied the partial differential equations of the course to attempt this.*]

4. Let $z(x,y)$ be a function defined implicitly by the equation:
$$x - \alpha z = \phi(y - \beta z),$$
where α, β are real constants, and ϕ is an arbitrary differentiable function. Show that z satisfies the equation:
$$\alpha\frac{\partial z}{\partial x} + \beta\frac{\partial z}{\partial y} = 1.$$
[*Hint: you can still use the normal single-variable chain rule here when taking each of the partial derivatives! Why?*]

5. Consider the function $u(x,y) = x\phi(y/x)$, where ϕ is a differentiable function of its argument and $x \neq 0$. Show that u satisfies:
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

6. If $u(x,y) = \phi(xy) + \sqrt{xy}\,\psi(y/x)$, where ϕ and ψ are twice-differentiable functions of their arguments, show that u satisfies the partial differential equation:
$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Introduction to differentials, and the multivariable chain rule

7. We define the *differential* df of the function $f(x, y, ...)$ to be the infinitesimal change in the function when its arguments are changed by an infinitesimal amount, $f(x + dx, y + dy, ...)$.

   (a) Let $f(x)$ be a function of a single variable. By Taylor expanding, derive the *single variable chain rule in differential form*:
   $$df = \frac{df}{dx}dx.$$

   (b) Let $f(x, y)$ be a function of two variables. By Taylor expanding twice, derive the *multivariable chain rule in differential form*:
   $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

8. Using the multivariable chain rule in differential form, find the differentials of the following functions:

   (a) $\exp(-1/(x + y))$, (b) $\sinh(x)/\sinh(y)$, (c) $\sqrt{x^2 + y^2}$, (d) $\arctan(y/x)$, (e) x^y.

9. Using both the single and multivariable chain rules in differential form, show that:

   (a) If $f \equiv f(x)$ is a function of a single variable x, and $x \equiv x(t)$ is a function of a single variable t, then:
   $$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

   (b) If $f \equiv f(x, y)$ is a function of two variables x, y, and $x \equiv x(t), y \equiv y(t)$ are both functions of a single variable t, then:
   $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

   (c) If $f \equiv f(x, y)$ is a function of two variables x, y, and $x \equiv x(u, v), y \equiv y(u, v)$ are both functions of two variables u, v, then:
   $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}.$$
   Derive a similar formula for $\partial f/\partial v$.

   (d) If $f \equiv f(x, y)$ is a function of two variables x, y, and $y \equiv y(x)$ is a function of the variable x, then:
   $$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$
   What is the difference between df/dx and $\partial f/\partial x$ here?

10. Using the multivariable chain rule, show that if $f(x, y) = x^2 + y^2$, and $x(u, v) = u^3 - 2v, y(u, v) = 3v - 2u^2$, we have:
    $$\frac{\partial f}{\partial u} = 2u\left(3u^4 - 6uv - 6v + 4u^2\right), \qquad \frac{\partial f}{\partial v} = 2\left(13v - 6u^2 - 2u^3\right).$$
    Check your results by writing f in terms of u, v first, then taking partial derivatives.

11. Using the multivariable chain rule, show that if $f(x, y) = xy$ and $y = x\cos(x)$, we have:
    $$\frac{df}{dx} = x\left(2\cos(x) - x\sin(x)\right).$$
    Check your result by writing f in terms of x first, then taking partial derivatives.

12. Let (x, y) be plane Cartesian coordinates, and let (r, θ) be plane polar coordinates. Let $f \equiv f(x, y)$ be a multivariable function whose expression in terms of Cartesian coordinates is $f(x, y) = e^{-xy}$.

    (a) Compute $\partial f/\partial x$ and $\partial f/\partial y$.

    (b) Compute $\partial f/\partial r$ and $\partial f/\partial \theta$, by: (i) writing f in terms of polar coordinates; (ii) using the multivariable chain rule.

The multivariable chain rule for second-order derivatives

13. Let $f(u,v) = u^2 \sinh(v)$, and let $u = x, v = x + y$.
    (a) By differentiating with respect to u, compute $\partial^2 f/\partial u^2$.
    (b) Using the multivariable chain rule, show that:
    $$\frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2},$$
    Hence compute the derivative in (a) by writing f in terms of x, y, differentiating, and using this relationship.
    (c) Repeat this exercise for the derivatives $\partial^2 f/\partial v^2$ and $\partial^2 f/\partial u \partial v$.

14. Let $f(u,v)$ be a multivariable function of $u(x,y) = 1 + x^2 + y^2$, $v(x,y) = 1 + x^2y^2$, where (x,y) are plane Cartesian coordinates.
    (a) Calculate $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\partial y$ in terms of the derivatives of f with respect to u, v.
    (b) For $f(u,v) = \log(uv)$, find $\partial^2 f/\partial x \partial y$ by: (i) using the expression derived in part (a); (ii) first expressing f in terms of x, y and then differentiating directly. Verify that your results agree.

15. Let (x,y) be plane Cartesian coordinates, and let (u,v) be plane Cartesian coordinates which are rotated an angle θ anticlockwise about the origin relative to the (x,y) coordinates. Let f be an arbitrary multivariable function of either (x,y) or (u,v). Show that:
    $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}.$$

16. Let (x,y) be plane Cartesian coordinates, and let (r,θ) be plane polar coordinates. Let f be a multivariable function. Show that:
    $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$
    Hence determine all solutions of the partial differential equation:
    $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$
    which are rotationally symmetric about the origin.

17. Consider a function $z(x,y)$ that satisfies $z(\lambda x, \lambda y) = \lambda^n z(x,y)$ for any real λ and a fixed integer n. Show that:
    $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz,$$
    and
    $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$