Part IA: Mathematics for Natural Sciences A
Examples Sheet 16: Scalar and vector fields, conservative vector fields, and line integrals

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Vector fields and vector derivatives

1. (a) Sketch the vector fields $\mathbf{G}_1 = (x, y, z)$ and $\mathbf{G}_2 = (y, -x, 0)$.
   (b) Calculate $\nabla \cdot \mathbf{G}_1$, $\nabla \cdot \mathbf{G}_2$ and $\nabla \times \mathbf{G}_1, \nabla \times \mathbf{G}_2$. Relate your results to the sketches you produced in part (a).

2. Find the divergence and curl of each of the following vector fields, where **a**, **b** are constant vectors:

   (a) $\mathbf{x}$, (b) $\mathbf{a}(\mathbf{x} \cdot \mathbf{b})$, (c) $\mathbf{a} \times \mathbf{x}$, (d) $\mathbf{x}/|\mathbf{x}|^3$, (e) $(\mathbf{a} \cdot \mathbf{b})\mathbf{x} + \mathbf{a}$, (f) $\mathbf{a} \times (\mathbf{b} \times \mathbf{x})$.

3. For each of the following pairs of vector fields, calculate $\nabla \cdot (\mathbf{F} \times \mathbf{G})$ and $\nabla \times (\mathbf{F} \times \mathbf{G})$:

   (a) $\mathbf{F} = (x, y, z), \mathbf{G} = (y, -x, 0)$;
   (b) $\mathbf{F} = (\sin(x), \sin(y), \sin(z)), \mathbf{G} = (\cos(x), \cos(y), \cos(z))$.

   Verify that the identities

   $$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \qquad \text{and} \qquad \nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

   hold in each case, where $(\mathbf{G} \cdot \nabla)\mathbf{F} = G_x \partial \mathbf{F}/\partial x + G_y \partial \mathbf{F}/\partial y + G_z \partial \mathbf{F}/\partial z$. [*We will learn how to efficiently prove these identities using 'suffix notation' in Part IB Mathematics.*]

Second-order derivatives

4. Define the *Laplacian* $\nabla^2 \phi$ of a scalar field ϕ. Evaluate the Laplacian of the scalar field $\phi = \frac{1}{2}(x^2 + y^2 + z^2)$.

5. Show that for any scalar field ϕ and any vector field **F**, we have $\nabla \times (\nabla \phi) = \mathbf{0}$ and $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Line integrals

6. Let $f(\mathbf{x})$ be a scalar field and let $\mathbf{F}(\mathbf{x})$ be a vector field. Suppose that $\gamma(t)$ is a curve. Give a summary of the steps required to evaluate the following types of line integrals:

   $$\int_\gamma f(\mathbf{x})\, |d\mathbf{x}|, \qquad \int_\gamma \mathbf{F} \cdot d\mathbf{x}, \qquad \int_\gamma \mathbf{F} \times d\mathbf{x}.$$

   Which of the integrals evaluate to scalars, and which evaluate to vectors?

7. By directly parametrising the curves, evaluate the following line integrals over the specified curves:

   $$\text{(a)} \int_{C_1} (x^2 + y^2)\, ds, \qquad \text{(b)} \int_{C_2} (x^2 + y^2)\, ds, \qquad \text{(c)} \int_{C_3} xy\, ds;$$

   where $ds = |d\mathbf{x}|$ is the infinitesimal arclength, and:

   - C_1 is a straight line from $(0, 0)$ to $(1, 1)$ in two dimensions;
   - C_2 is the unit circle centred on the origin in two dimensions;
   - C_3 is a helix, $x(t) = \cos(6t), y(t) = \sin(6t), z(t) = 8t$ in three dimension, with $0 \leq t \leq \pi/12$.

   Give a sketch of the paths of integration in each case.

8. By directly parametrising the curves, evaluate the following line integrals over the specified curves:

$$\text{(a)} \int\limits_{C_1} \begin{pmatrix} y \\ -x \\ -1 \end{pmatrix} \cdot d\mathbf{x}, \qquad \text{(b)} \int\limits_{C_2} \begin{pmatrix} xy \\ x^2+y \\ 0 \end{pmatrix} \cdot d\mathbf{x}, \qquad \text{(c)} \int\limits_{C_3} \begin{pmatrix} xy \\ x^2+y \\ 0 \end{pmatrix} \cdot d\mathbf{x};$$

where:

- C_1 is the helix $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$, with $0 \leq t \leq 2\pi$.
- C_2 is the planar curve $x(t) = t, y(t) = t^2, z(t) = 0$, with $0 \leq t \leq 1$;
- C_3 is the planar curve $x(t) = 2t - t^2, y(t) = t^4 - 4t^3 + 4t^2, z(t) = 0$, with $0 \leq t \leq 1$;

Give a sketch of the paths of integration in each case.

9. By directly parametrising the curve, evaluate each of the line integrals:

$$\int\limits_C d\mathbf{x}, \qquad \int\limits_C \mathbf{x} \times d\mathbf{x}, \qquad \int\limits_C \mathbf{x} \times (\mathbf{x} \times d\mathbf{x}),$$

where C is the unit circle centred on the origin in the x-y plane, traversed anticlockwise about the z-axis. How does the result of the second integral relate to the vector area of the unit disk centred on the origin in the x-y plane?

Conservative vector fields

10. State what it means for a vector field $\mathbf{F}$ to be (a) *conservative*; (b) *irrotational*. How are these conditions related?

11. Show that the following vector fields are conservative by finding potential functions in each case:

$$\text{(a) } \mathbf{F}_1 = (yz, xz, xy), \qquad \text{(b) } \mathbf{F}_2 = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Now, check directly that each of these vector fields is irrotational. $(*)$ If you are doing Part IA Physics, what is the physical relevance of the second example?

12. Two vector fields are given by $\mathbf{F}(\mathbf{r}) = \mathbf{a} \times \mathbf{r}$ and $\mathbf{G} = f(|\mathbf{r}|)\mathbf{r}$, where $\mathbf{a}$ is a constant vector, $\mathbf{r} = (x, y, z)$ is the general position vector, and f is an arbitrary scalar function. Which, if any, of these functions can be written as the gradient of a scalar function? For those that can, find an appropriate scalar function.

13. Show that $\mathbf{F}$ is conservative if and only if $\mathbf{F} \cdot d\mathbf{x}$ is an exact differential, where $d\mathbf{x} = (dx, dy, dz)$ is an infinitesimal displacement vector.

Line integrals of conservative vector fields

14. State and prove the *gradient theorem* for line integrals. Using the gradient theorem, show that if $\mathbf{F}$ is conservative, and C_1, C_2 are paths of integration with the same start and end points, then:

$$\int\limits_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int\limits_{C_2} \mathbf{F} \cdot d\mathbf{x}.$$

Deduce also that the line integral of a conservative field around a closed loop is always zero.

15. (a) Without using the gradient theorem, evaluate the integral of the vector field:

$$\mathbf{F}(\mathbf{r}) = e^{-r^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\mathbf{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$, along the curve:

$$\mathbf{r}(t) = \begin{pmatrix} \cos(2\pi t^n) \\ \sin(2\pi t^n) \\ 1 \end{pmatrix},$$

for $0 \leq t \leq 1$, and where n is a fixed positive integer.

(b) Now, show that **F** is conservative. Hence, use the gradient theorem to verify your answer in part (a) is correct.

16. Consider the following line integrals:

$$\int_{\gamma_i} \mathbf{F} \cdot d\mathbf{x}, \qquad \int_{\gamma_i} \mathbf{G} \cdot d\mathbf{x},$$

where:

$$\mathbf{F} = \begin{pmatrix} 4x - 2y - 2z \\ -2x + 2y + az \\ -2x + ay + 2z \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} y\cos(xy) - by + (b+c)z \\ x\cos(xy) + bx - bz \\ (c-b)x + by \end{pmatrix},$$

where a, b and c are real constants, and:

- γ_1 is the set of straight lines connecting the points $(0,0,0)$ to $(1,1,0)$, followed by $(1,1,0)$ to $(1,1,1)$;
- γ_2 is the curve from $(0,0,0)$ to $(1,1,1)$ on which the position vector is parametrised by $\mathbf{x}(t) = (t, t, t^2)$ for $0 \leq t \leq 1$.

For each value of a, b, c:

(a) Evaluate each of the line integrals by directly parametrising the paths of integration.

(b) Decide whether **F**, **G** are conservative vector fields, and relate this result to the calculation you performed in part (a).