

Part IA: Mathematics for Natural Sciences A

Examples Sheet 17: Surface integrals and integral theorems

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Surface integrals in cylindrical, spherical, and Cartesian coordinates

- (Infinitesimal vector areas)** For each of the following surfaces, draw convincing sketches showing that (for some appropriate choices of orientation of the surfaces):
 - On the flat circular surface on the top of a cylinder of radius a , oriented along the z -axis, we have $d\mathbf{S} = \hat{\mathbf{n}}rdrd\theta$, where $\hat{\mathbf{n}} = (0, 0, 1)$.
 - On the curved surface of a cylinder of radius a , oriented along the z -axis, we have $d\mathbf{S} = \hat{\mathbf{n}}ad\theta dz$, where $\hat{\mathbf{n}} = (\cos(\theta), \sin(\theta), 0)$.
 - On a sphere of radius a , centred at the origin, we have $d\mathbf{S} = \hat{\mathbf{n}}a^2 \sin(\theta)d\theta d\phi$, where $\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$.
 - On a plane $\mathbf{r} \cdot \hat{\mathbf{n}} = d$, where $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ is a constant unit normal, we have:

$$d\mathbf{S} = \frac{\hat{\mathbf{n}}}{n_z} dx dy = \frac{\hat{\mathbf{n}}}{n_y} dx dz = \frac{\hat{\mathbf{n}}}{n_x} dy dz,$$

depending on whether we choose to parametrise the plane with the coordinates (x, y) , (x, z) or (y, z) (noting that when one of the components of $\hat{\mathbf{n}}$ vanishes, such a parametrisation may be impossible).

How are these infinitesimal vector areas related to the infinitesimal scalar areas, dS ?

- (Some cylindrical surface integrals)** Evaluate the following surface integrals directly, without using the divergence theorem. Here, $\hat{\mathbf{n}}$ is the unit normal to the surface.

$$(a) \int_S \hat{\mathbf{n}} \cdot d\mathbf{S}, \quad (b) \int_S xy dS, \quad (c) \int_S \begin{pmatrix} -yz \\ x^2 \\ 0 \end{pmatrix} \cdot d\mathbf{S}, \quad (d) \int_S \begin{pmatrix} y^2 \\ x^2 \\ 0 \end{pmatrix} \cdot d\mathbf{S},$$

where:

- S is the unit disc in the $z = 0$ plane, centred on the origin (take the normal in the positive z -direction);
 - S is the curved surface of the cylinder described by $0 \leq z \leq 1$, $0 \leq \theta \leq 2\pi$, $r = 1$, in cylindrical polar coordinates (taking the normal pointing in the positive r -direction).
- (Some spherical surface integrals)** Evaluate the following surface integrals directly, without using the divergence theorem. In each case, S is the surface of the unit sphere centred at the origin, $\hat{\mathbf{n}}$ is the outward-pointing unit normal, and $\mathbf{r} = (x, y, z)$ is the standard Cartesian position vector.

$$(a) \int_S \hat{\mathbf{n}} \cdot d\mathbf{S}, \quad (b) \int_S xy dS, \quad (c) \int_S \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \cdot d\mathbf{S}, \quad (d) \int_S \frac{\sin(\theta)\mathbf{r}}{r} \cdot d\mathbf{S}, \quad (e) \int_S \nabla \times \nabla(x^2 + y^2 + z^2) \cdot d\mathbf{S}.$$

- (A composite shape)** Without using the divergence theorem, evaluate:

$$\int_S (z + y^3) dS,$$

where S is the total surface made from the vertical cylinder $x^2 + y^2 = a^2$ with $0 \leq z \leq b$, the flat disc $x^2 + y^2 \leq a^2$ in the $z = b$ plane ($b > a > 0$), and the hemispherical indentation $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$.

5. **(Some Cartesian surface integrals)** Evaluate the following surface integrals directly, without using the divergence theorem:

$$(a) \int_{S_1} \begin{pmatrix} e^{-x} \\ x^{-1}(\log^2(x) + 1)^{-1} \\ z \end{pmatrix} \cdot d\mathbf{S}, \quad (b) \int_{S_2} \begin{pmatrix} |x| \\ 2xy \\ e^{-z} \end{pmatrix} \cdot d\mathbf{S}, \quad (c) \int_{S_3} (x + yz) dS, \quad (d) \int_{S_4} \begin{pmatrix} x + y + z \\ (x + y + z)^2 \\ 1 - x - y - z \end{pmatrix} \cdot d\mathbf{S}$$

where:

- S_1 is the surface of an axes-aligned unit cube, two of whose vertices are at $(0, 0, 0)$ and $(1, 1, 1)$;
 - S_2 is the surface of an axes-aligned unit cube, two of whose vertices are at $(1, 0, 0)$ and $(2, 1, 1)$;
 - S_3 is the surface $0 \leq x \leq 2\pi, 0 \leq z \leq 1$ and $y = 1 + z$;
 - S_4 is the surface of the tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0),$ and $(0, 0, 1)$.
6. **(A more abstract surface integral)** Let \mathbf{a}, \mathbf{b} be constants, and let $\mathbf{r} = (x, y, z)$ be the standard Cartesian position vector. Without using the divergence theorem, calculate the flux of the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{r}$ through:
- (a) the triangle OAB where O denotes the origin (where A has position vector \mathbf{a} and B has position vector \mathbf{b});
 - (b) the closed hemisphere with curved surface and base given by $x^2 + y^2 + z^2 = R^2, 0 \leq z \leq R$ and $x^2 + y^2 \leq R^2, z = 0$, respectively, where the parameter $R > 0$.

Surface integrals in general coordinate systems

7. Suppose that a surface S is described as the set of all position vectors $\mathbf{r}(u, v)$ as the parameters u, v vary. Show that the infinitesimal vector area on the surface near the point with parameters (u, v) is given by:

$$d\mathbf{S} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv.$$

Hence, rederive the formulae obtained in Question 1 algebraically.

8. Consider the semi-infinite cone described by the equation $bz = ab - a\sqrt{x^2 + y^2}$.
- (a) Show that the infinitesimal vector area element on the surface of the cone may be expressed equivalently in terms of Cartesian coordinates (x, y) or cylindrical polar coordinates (r, θ) via:

$$d\mathbf{S} = \begin{pmatrix} \frac{ax}{b\sqrt{x^2+y^2}} \\ \frac{ay}{b\sqrt{x^2+y^2}} \\ 1 \end{pmatrix} dxdy = \begin{pmatrix} a \cos(\theta)/b \\ a \sin(\theta)/b \\ 1 \end{pmatrix} r dr d\theta.$$

- (b) Hence, evaluate the surface area of a cone of height a , with base of radius b .
 - (c) Find also the flux of the vector field $\mathbf{F} = (y, x, 1)$ through the curved surface of the cone with $z \geq 0$.
9. Consider the paraboloid described by the equation $b^2z = ab^2 - ax^2 - ay^2$.

- (a) Show that the infinitesimal vector area element on the surface of the paraboloid may be expressed equivalently in terms of Cartesian coordinates (x, y) or cylindrical polar coordinates (r, θ) via:

$$d\mathbf{S} = \begin{pmatrix} 2ax/b^2 \\ 2ay/b^2 \\ 1 \end{pmatrix} dxdy = \begin{pmatrix} 2ar \cos(\theta)/b^2 \\ 2ar \sin(\theta)/b^2 \\ 1 \end{pmatrix} r dr d\theta.$$

- (b) Hence, evaluate the surface area of a paraboloid of height a , with base of radius b .
- (c) Find also the flux of the vector field $\mathbf{F} = (y, x, 1)$ through the curved surface of the paraboloid with $z \geq 0$.

The divergence theorem

10. Carefully state the *divergence theorem*. You should define all symbols that arise and specify orientations carefully.
11. Check explicitly that the divergence theorem holds for:
- the flux of the vector field $\mathbf{F} = (x^2 + y^2, 3xy, 6z)$ through the surface of the unit cube $0 \leq x \leq 1, 0 \leq y \leq 1$ and $0 \leq z \leq 1$;
 - the flux of the vector field $\mathbf{F} = (x^3, y^3, z^3)$ through the surface of the unit sphere centred on the origin;
 - the flux of the vector field $\mathbf{F} = (0, (y+2x-4)^2, 1-z^2)$ through the surface of the triangular prism, $0 \leq z \leq 1$, whose base in the xy -plane has vertices $(0, 0, 0), (2, 0, 0), (0, 4, 0)$.
12. Evaluate: (a) $\int_S \mathbf{r} \cdot d\mathbf{S}$, (b) $\int_V \nabla \cdot \hat{\mathbf{r}} dV$, where V is the unit sphere, S is the surface of the unit sphere, and $\mathbf{r} = (x, y, z)$ is the standard Cartesian position vector.

13. Evaluate:

$$\int_S \begin{pmatrix} 0 \\ (y+3)^2 \\ z^2 \end{pmatrix} \cdot d\mathbf{S},$$

where S is the surface of the pyramid bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z/2 = 1$.

14. (a) Starting from the Maxwell equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, where $\mathbf{E}(\mathbf{x}, t)$ is the electric field, $\rho(\mathbf{x}, t)$ is charge density, and ϵ_0 is a constant called the *permittivity of free space*, show that $\Phi_E = Q/\epsilon_0$, where Φ_E is the electric flux through a closed surface S , and Q is the total charge contained in the volume bounded by that surface. This result is called *Gauss' law for electric fields*.
- (b) Another of the Maxwell equations states that $\nabla \cdot \mathbf{B} = 0$, where $\mathbf{B}(\mathbf{x}, t)$ is the magnetic field. What does the divergence theorem tell you about the magnetic flux through a closed surface S ?

Stokes' theorem

15. Carefully state *Stokes' theorem*. You should define all symbols that arise and specify orientations carefully.
16. Check explicitly that Stokes' theorem holds for:
- the surface integral of the curl of $\mathbf{F} = (yz, zx, xy)$ through the disk $x^2 + y^2 \leq 1, z = 0$;
 - the surface integral of the curl of $\mathbf{F} = (y, z, x)$ through the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$;
 - the surface integral of the curl of $\mathbf{F} = (yz, zx, xy)$ through the paraboloid $z = 4 - x^2 - y^2, z \geq 0$.
17. Using Stokes' theorem, evaluate the line integral of $\mathbf{F} = (z, x, y)$ over the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, oriented anticlockwise as seen from above.
18. Starting from the Maxwell equation $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, where $\mathbf{E}(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t)$ are the electric and magnetic field respectively, show that $\mathcal{E} = -d\Phi_B/dt$, where Φ_B is the magnetic flux through a time-independent surface S , and \mathcal{E} is the *electromotive force* around the boundary C , which you should find an expression for in terms of a line integral. This result is called *Faraday's law of induction*.