Part IA: Mathematics for Natural Sciences A
Examples Sheet 3: Vector area, polar coordinate systems, and complex numbers

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Vector area

1. (a) Define the *vector area* **A** of a surface composed of k flat faces with areas $A_1, ..., A_k$ and unit normals $\hat{\mathbf{n}}_1, ..., \hat{\mathbf{n}}_k$. What are the conventions usually used when choosing the unit normal(s)?

   (b) In terms of the position vectors **a**, **b** determine the vector areas of: (i) the parallelogram defined by **a**, **b**; (ii) the triangle defined by **a**, **b**. Hence, given points $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,1,1)$, $C = (0,2,0)$, compute the vector area of the triangle OAB (with vertices taken in that order), and the vector area of the loop $OABC$ composed of straight-line segments (with vertices taken in that order).

2. (a) Give a very general explanation of how the idea of vector area could be extended to *curved surfaces*, and hence explain why we expect the vector area of any *closed* surface to be **0**.

   (b) Compute the vector area of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, taken in that order. Hence, compute the vector area of the pyramid extending this square with the point $(1,1,1)$, excluding its square face.

   (c) Compute the vector area of a truncated hollow cone, bounded by a horizontal circle of radius 4 units and a horizontal circle of radius 3 units at some height above the first (note the result is independent of the height!).

3. (a) Let **S** be the vector area of the surface S. Prove that the area of the projection of the surface S onto the plane with unit normal $\hat{\mathbf{m}}$ is $|\mathbf{S} \cdot \hat{\mathbf{m}}|$. [*Hint: consider joining the surface to its 'shadow' on the plane to create a closed surface.*]

   (b) Compute the vector area of the projection of the square with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$ onto the plane with unit normal $\hat{\mathbf{m}} = (0,-1,1)/\sqrt{2}$.

   (c) By projecting areas onto the yz, xz, and xy planes, compute the vector area of the loop with vertices $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,1,1)$, $C = (0,2,0)$, taken in that order. [*Your answer should match your answer to Question 1(b)!*] What is the area of the loop projected onto: (i) the plane with normal $(0,-1,1)$; (ii) the plane that maximises the projected area?

Polar coordinate systems

4. Draw (convincing) diagrams defining plane, cylindrical, and spherical polar coordinates. In each case, derive the coordinate transform laws from polars to Cartesians, and from Cartesians to polars. Hence, find the cylindrical polar and spherical polar coordinates of the point $(3,4,5)$.

5. (a) In 2D Cartesian coordinates, a circle is specified by $(x-1)^2 + y^2 = 1$. Find its equation in polar coordinates.

   (b) In 3D Cartesian coordinates, a sphere is specified by $(x-1)^2 + y^2 + z^2 = 1$. Find its equation in spherical polar coordinates.

6. Let $a > 0$ be a constant. Describe the following loci:

   (a) (i) $\phi = a$; (ii) $r = \phi$, in plane polar coordinates.

   (b) (i) $z = a$; (ii) $r = a$; (iii) $r = a$ and $z = \phi$, in cylindrical polar coordinates.

   (c) (i) $\theta = a$; (ii) $\phi = a$; (iii) $r = a$; (iv) $r = \theta = a$, in spherical polar coordinates.

7. Consider a point with position vector $\hat{\mathbf{n}}$ on the unit sphere S.

   (a) Explain why $\hat{\mathbf{n}} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, where θ, ϕ are the spherical coordinates of $\hat{\mathbf{n}}$.

   (b) Show that the vector area $d\mathbf{S}$ of a small patch near $\hat{\mathbf{n}}$, subtending a small angle $d\theta$ in the θ direction, and a small angle $d\phi$ in the ϕ direction, is given *approximately* by $d\mathbf{S} = \hat{\mathbf{n}} \sin(\theta) d\theta d\phi$. Why might this be useful?

Real and imaginary parts

8. Find the real and imaginary parts of the following numbers (where n is an integer):

   (a) i^3, (b) i^{4n}, (c) $\left(\frac{1+i}{\sqrt{2}}\right)^2$, (d) $\left(\frac{1-i}{\sqrt{2}}\right)^2$, (e) $\left(\frac{1+\sqrt{3}i}{2}\right)^3$, (f) $\frac{1+i}{2-5i}$, (g) $\left(\frac{1+i}{1-i}\right)^2$.

9. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:

   (a) z^2, (b) iz, (c) $(1+i)z$, (d) $z^2(z-1)$, (e) $z^*(z^2 - zz^*)$.

10. Define u and v to be the real and imaginary parts, respectively, of the complex function $w = 1/z$. Show that the contours of constant u and v are circles. Show also that the contours of u and the contours of v intersect at right angles.

Factoring polynomials and solving equations

11. Factorise the following expressions: (a) $z^2 + 1$; (b) $z^2 - 2z + 2$; (c) $z^2 + i$; (d) $z^2 + (1-i)z - i$. *[Hint: you have already computed the two square roots of i in Question 8(c).]*

12. Given that $z = 2 + i$ solves the equation $z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0$, find the remaining solutions.

13. Consider the polynomial equation $a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0$, where the coefficients $a_n, a_{n-1}, ..., a_0$ are real. Show that the solutions to this equation come in complex conjugate pairs. Deduce that if n is odd, there is at least one real solution.

Geometry of complex numbers

14. Using a diagram, explain the geometric meaning of the *modulus*, $|z|$, and *argument*, $\arg(z)$, of a complex number z. Find the moduli and (principal) arguments of: (a) $1 + \sqrt{3}i$; (b) $-1 + i$; (c) $-\sqrt{3} - i/\sqrt{3}$.

15. For $z \in \mathbb{C}$, show that $|z|^2 = zz^*$. Hence prove that $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$, where $a, b \in \mathbb{C}$, and interpret this result geometrically. *[Hint: you don't need to split a, b into real and imaginary parts.]*

16. By writing $z = |z|(\cos(\arg(z)) + i\sin(\arg(z)), w = |w|(\cos(\arg(w)) + i\sin(\arg(w))$, compute the modulus and argument of the product zw. Hence give the geometrical interpretation of multiplying one complex number by another complex number. Give also a geometrical interpretation of division of one complex number by another complex number, z/w.

17. Let $z_1 = 2 + i$, $z_2 = 3 + 4i$. Find $z_1 z_2$ by: (a) adding arguments and multiplying moduli; (b) using the rules of complex algebra. Verify that your results agree.

18. By considering multiplication of the complex numbers $z = 1 + iA$ and $w = 1 + iB$, derive the arctangent addition formula:
$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right).$$

19. Give a geometrical interpretation (in terms of *vectors*) of the real and imaginary parts of the quantity $Q = z_1 z_2^*$. Show also that Q is invariant under a rotation of z_1, z_2 about the origin, and confirm that this is consistent with your geometrical interpretation. *[Hint: In Question 16, you showed that multiplying by a complex number u of unit modulus is equivalent to a rotation about the origin.]*