Part IA: Mathematics for Natural Sciences A
Examples Sheet 4: More complex numbers, and hyperbolic functions

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Loci in the complex plane

1. (**Circles**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

   (a) $|z| = 4$, (b) $|z-1| = 3$, (c) $|z-i| = 2$, (d) $|z-(1-2i)| = 3$, (e) $|z^*-1| = 1$, (f) $|z^*-i| = 1$.

2. (**Transformations of circles**) Describe the set of points $z \in \mathbb{C}$ satisfying $|z-2-i| = 6$. Without further calculation, describe the sets of points $u \in \mathbb{C}$, $v \in \mathbb{C}$, $w \in \mathbb{C}$ satisfying:

   (a) $u = z + 5 - 8i$, (b) $v = iz + 2$, (c) $w = \frac{3}{2}z + \frac{1}{2}z^*$,

   where $|z-2-i| = 6$.

3. (**Circles of Apollonius**) Let $a, b \in \mathbb{C}$. Show that the set of points satisfying $|z-a| = \lambda|z-b|$, where $\lambda \neq 1$, is a circle in the complex plane. *[Hint: start by squaring the equation. You don't need to split z into real and imaginary parts.]* Determine the centre and radius of the circle $|z| = 2|z-2|$.

4. (**Lines and half-lines**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

   (a) $|z-2| = |z+i|$, (b) $|z-2| = |z^*+i|$, (c) $\arg(z) = \pi/2$, (d) $\arg(z^*) = \pi/4$.

5. (**Lines and circles**) Let $a \in \mathbb{R}$ and $b, c \in \mathbb{C}$. *Without setting $z = x + iy$*, describe the locus $azz^* + bz + b^*z^* + c = 0$ for different values of a, b, c. How does the locus change under the maps: (a) $z \mapsto \alpha z$ for $\alpha \in \mathbb{C}$; (b) $z \mapsto 1/z$?

6. (**More complex figures**) Sketch the sets of points $z \in \mathbb{C}$ satisfying:

   (a) $\text{Re}(z^2) = \text{Im}(z^2)$, (b) $\dfrac{\text{Im}(z^2)}{z^2} = -i$, (c) $|z^*+2i| + |z| = 4$, (d) $|2z - z^* - 3i| = 2$.

Exponential form of a complex number

7. State *Euler's formula* for the complex exponential $e^{i\theta}$. Hence provide a simpler derivation of the modulus-argument multiplication law proved in Question 16 of Sheet 3.

8. Find (a) the real and imaginary parts; (b) the modulus and argument, of:

$$\frac{e^{i\omega t}}{R + i\omega L + (i\omega C)^{-1}},$$

   where ω, t, R, L, C are real, quoting your answers in terms of $X = \omega L - (\omega C)^{-1}$.

9. Express each of the following in Cartesian form: (a) $e^{-i\pi/2}$; (b) $e^{-i\pi}$; (c) $e^{i\pi/4}$; (d) e^{1+i}; (e) $e^{2e^{i\pi/4}}$.

10. Let a, b, ω be real constants. Show that $a\cos(\omega x) + b\sin(\omega x) = \text{Re}((a-bi)e^{i\omega x})$, and hence, by writing $a - bi$ in exponential form, deduce that $a\cos(\omega x) + b\sin(\omega x) = \sqrt{a^2+b^2}\cos(\omega x - \arctan(b/a))$.

Multi-valued functions: logarithms and powers

11. Explain why the complex logarithm $\log : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is a *multi-valued function*, and give its possible values. Using the complex logarithm, find all complex numbers satisfying: (a) $e^{2z} = -1$; (b) $e^{z^*} = i + 1$.

12. Let the real and imaginary parts of the complex logarithm $\log(z)$ be u, v respectively. Sketch the contours of constant u, v in the complex plane, and show that they intersect at right angles.

13. Find the real and imaginary parts of the function $f(z) = \log(z^{1+i})$. Hence, sketch the locus $\mathrm{Re}(f(z)) = 0$.

14. Explain how the complex logarithm can be used to define complex powers, z^w, and hence describe the multi-valued nature of complex exponentiation. Compute all values of the multi-valued exponentials: (a) i^i; (b) $i^{1/3}$.

15. Compute all possible values of $\left(i^i\right)^i$ and $i^{\left(i^i\right)}$.

Roots of unity

16. Write down the solutions to the equation $z^n = 1$ in terms of complex exponentials, and plot the solutions on an Argand diagram. *[Recall that the solutions are called the* nth *roots of unity.]*

17. Find and plot the solutions to the following equations: (a) $z^3 = -1$; (b) $z^4 = 1$; (c) $z^2 = i$; (d) $z^3 = -i$.

18. If $\omega^n = 1$, determine the possible values of $1 + \omega + \omega^2 + \cdots + \omega^{n-1}$, and interpret your result geometrically.

19. Show that the roots of the equation $z^{2n} - 2bz^n + c = 0$ will, for general complex values of b and c and integral values of n, lie on two circles in the Argand diagram. Give a condition on b and c such that the circles coincide. Find the largest possible value for $|z_1 - z_2|$, if z_1 and z_2 are roots of $z^6 - 2z^3 + 2 = 0$.

Trigonometry with complex numbers

20. Prove *De Moivre's formula*, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$. Hence, solve the equation $16\sin^5(\theta) = \sin(5\theta)$ by expressing $\sin(5\theta)$ in terms of $\sin(\theta)$ and its powers.

21. Starting from Euler's formula, show that the trigonometric functions can be written in terms of complex exponentials as:
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$
Learn these formulae off by heart. Hence, express $\sin^5(\theta)$ in terms of $\sin(\theta), \sin(3\theta)$ and $\sin(5\theta)$.

22. Show that if $x, y \in \mathbb{R}$, the equation $\cos(y) = x$ has the solutions $y = \pm i \log\left(x + i\sqrt{1 - x^2}\right) + 2n\pi$ for integer n.

23. Find the real and imaginary parts of the function $\tan(z^*)$.

24. Let $\theta \neq 2p\pi$ for $p \in \mathbb{Z}$. Show that $\displaystyle\sum_{n=0}^{N-1} \cos(n\theta) = \frac{\cos\left((N-1)\theta/2\right)\sin\left(N\theta/2\right)}{\sin\left(\theta/2\right)}$. What happens if $\theta = 2p\pi$?

Hyperbolic functions

25. (a) Give the definitions of $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

    (b) Hence, show that $\cos(x) = \cosh(ix)$ and $i\sin(x) = \sinh(ix)$. Deduce *Osborn's rule*: 'a hyperbolic trigonometric identity can be deduced from a circular trigonometric identity[1] by replacing each trigonometric function with its hyperbolic counterpart *except* where sine enters quadratically, where we include an extra factor of -1.'

    (c) Using Osborn's rule, write down the formula for $\tanh(x+y)$ in terms of $\tanh(x), \tanh(y)$.

26. Find the real and imaginary parts of the following complex numbers:

    (a) $\log\left[\sinh\left(\frac{i\pi}{2}\right) + \cosh\left(\frac{9i\pi}{2}\right)\right]$, (b) $\sum_{n=1}^{121}\left[\tanh\left(\frac{in\pi}{4}\right) - \tanh\left(\frac{in\pi}{4} - \frac{i\pi}{4}\right)\right]$.

27. Let $b \geq a > 0$ be fixed, and let θ be a variable parameter. Find the Cartesian equations of the two parametric curves: (a) $(x, y) = (a\cos(\theta), b\sin(\theta))$; (b) $(x, y) = (a\cosh(\theta), b\sinh(\theta))$, and sketch them in the plane. [*This explains why hyperbolic functions are called hyperbolic functions!*]

28. Sketch the graphs of $\cosh(x), \sinh(x)$ and $\tanh(x)$, noting any asymptotes. Hence, sketch the graphs of $\cosh^{-1}(x)$, $\sinh^{-1}(x)$ and $\tanh^{-1}(x)$.

29. Express $\cosh^{-1}(x), \sinh^{-1}(x)$ and $\tanh^{-1}(x)$ as logarithms, justifying any sign choices you make.

30. Solve the equation $\cosh(x) = \sinh(x) + 2\text{sech}(x)$, giving the solutions as logarithms.

31. Find all solutions to the equations: (a) $\cosh(z) = i$; (b) $\sinh(z) = -2$; (c) $\tanh(z) = -i$.

[1] Provided the arguments of all the circular trigonometric functions are homogeneous linear polynomials in the variables of interest.