

Part IA: Mathematics for Natural Sciences A

Examples Sheet 5: Differential calculus, Riemann sums, and basic integrals

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Limit definition of the derivative

- Let $y \equiv y(x)$ be a function of x . Define the *derivative* dy/dx of y as a limit. Using the limit definition:
 - show that differentiation is a linear operation;
 - find the derivative of $y(x) = x^n$, for $n = 0, 1, 2, 3, \dots$Hence obtain the derivative of $ax + bx^2 \sin(\theta)$, where a, b, θ are real constants.
- Using *only* the limit definition, show that for $a > 0$, the derivative of $y(x) = a^x$ is proportional to a^x .
 - One definition of the number e is the value of a for which the proportionality constant in the previous part is 1. Using *only* this definition, show that the derivative of a^x is given by $\log(a)a^x$.

Rules of differentiation

- State the chain rule, the product rule, the quotient rule, and the reciprocal rule, respectively. Make sure you know them off by heart!
- Using the rules you derived in the previous question, compute the derivatives of:
 - $\log(x)$,
 - 3^{x^2} ,
 - $\frac{e^x}{x^3 - 1}$,
 - $x^3 \log(x^2 - 7)$,
 - $\sqrt{x^3 - e^x \log(x)}$.
- By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives: (a) $\cos(x)$; (b) $\sin(x)$; (c) $\cosh(x)$; (d) $\sinh(x)$; (e) $\tan(x)$; (f) $\tanh(x)$. Learn these derivatives off by heart.
- Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived on Sheet 4; (ii) the reciprocal rule, compute the derivatives of: (a) $\cosh^{-1}(x)$; (b) $\sinh^{-1}(x)$; (c) $\tanh^{-1}(x)$. Learn these derivatives off by heart.
- If $y \equiv y(x)$ is a function of x , show that $\frac{d^3 x}{dy^3} = - \left(\frac{dy}{dx} \right)^{-4} \frac{d^3 y}{dx^3} + 3 \left(\frac{dy}{dx} \right)^{-5} \left(\frac{d^2 y}{dx^2} \right)^2$. Verify this when $y = e^{2x}$.
- What is *implicit differentiation*, and why is it called implicit? Using: (a) implicit differentiation; (b) the reciprocal rule, find dy/dx given $y + e^y \sin(y) = 1/x$, and make sure that your answers agree.

Curve-sketching

- State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:
 - x ,
 - $\sin(x)$,
 - e^x ,
 - $\sin(\frac{\pi}{2} - x)$,
 - $|x| \cos(x)$,
 - \sqrt{x} ,
 - 2,
 - 0,
 - $\log \left| \frac{1+x}{1-x} \right|$.
- Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!
- Sketch the graphs of the following functions, explaining your reasoning in each case:

(a) $(x-3)^3 + 2x$, (b) $\frac{x}{1+x^2}$, (c) $\frac{x^2+3}{x-1}$, (d) xe^x , (e) $\frac{\log(x)}{1+x}$, (f) $\frac{1}{1-e^x}$, (g) $e^x \cos(x)$.

Riemann sums and the definition of the integral

12. Explain what is meant by a *Riemann sum* for a function $f : [a, b] \rightarrow \mathbb{R}$ using a *partition* $P = (x_0, \dots, x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, \dots, t_n)$. By choosing appropriate partitions and taggings in each case, use sequences of Riemann sums to evaluate the definite integrals of the following functions on $[0, 1]$ from first principles:

(a) x , (b) x^2 , (c) x^3 , (d) \sqrt{x} , (e) $\cos(x)$.

[Hint: For part (d), consider a non-uniform tagging. For part (e), consider the integral of $\operatorname{Re}(e^{ix})$ instead of $\cos(x)$.]

13. Using a non-uniform tagging, use a sequence of Riemann sums to evaluate the integral $\int_1^\infty \frac{dx}{x^{1+\alpha}}$, where $\alpha > 0$.

14. Assuming standard integrals, show by considering Riemann sums that $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{n^2 - k^2}}{n^2} = \frac{\pi}{4}$.

Basic integrals

15. Write down the indefinite integrals of each of the following functions, where $a \neq 0, \alpha \neq -1$, and f is any (differentiable, non-zero) function:

(a) $(ax + b)^\alpha$, (b) e^{ax+b} , (c) $(ax + b)^{-1}$, (d) $\sin(ax + b)$, (e) $\cos(ax + b)$,
(f) $\sec^2(ax + b)$, (g) $\operatorname{cosec}^2(ax + b)$, (h) $\sinh(ax + b)$, (i) $\cosh(ax + b)$, (j) $f'(x)f(x)^\alpha$,
(k) $f'(x)/f(x)$.

Learn these integrals off by heart, and get your supervision partner to test you on them.

16. Using the results of the previous question, evaluate the definite integrals:

(a) $\int_0^2 (x-1)^2 dx$, (b) $\int_0^\pi e^{i\theta} d\theta$, (c) $\int_0^\pi \cos(x) dx$, (d) $\int_{-\pi/4}^{\pi/4} \sec^2(x) dx$, (e) $\int_0^1 \frac{2x+4}{x^2+4x+1} dx$.

17. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax} \cos(bx)$. Similarly, determine the indefinite integrals of $e^x(\sin(x) - \cos(x))$ and $e^x(\sin(x) + \cos(x))$.