Part IA: Mathematics for Natural Sciences A
Examples Sheet 6: Methods of integration

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Integration by substitution

1. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

   (a) $\dfrac{1}{\sqrt{1-x^2}}$, (b) $\dfrac{1}{\sqrt{x^2-1}}$, (c) $\dfrac{1}{\sqrt{1+x^2}}$, (d) $\dfrac{1}{1+x^2}$, (e) $\dfrac{1}{1-x^2}$

   Learn these integrals off by heart, and get your supervision partner to test you on them.

2. Using the results of the previous question, determine: (a) $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}}$; (b) $\displaystyle\int \frac{8-2x}{\sqrt{6x-x^2}}\,dx$.

3. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

   (a) $x\sqrt{x+3}$, (b) $\tan(x)\sqrt{\sec(x)}$, (c) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$, (d) $\dfrac{1}{x\sqrt{x^2-1}}$.

4. This question shows that any trigonometric integral can be turned into an algebraic integral through the use of the powerful *half-tangent substitution*.

   (a) Show that if $t=\tan\left(\frac{1}{2}x\right)$, then $\sin(x)=2t/(1+t^2)$, $\cos(x)=(1-t^2)/(1+t^2)$ and $dx/dt=2/(1+t^2)$. Deduce that for any function f, we have:

   $$\int f\left(\sin(x),\cos(x)\right)\,dx = \int f\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

   (b) Using the method derived in (a), find the indefinite integrals of the following functions:

   (i) $\operatorname{cosec}(x)$, (ii) $\sec(x)$, (iii) $\dfrac{1}{2+\cos(x)}$.

Partial fractions and rational functions

5. Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

   (a) $\dfrac{1}{1-x^2}$, (b) $\dfrac{3x}{2x^2+x-1}$, (c) $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}$.

   Compare your answer to (a) with your answer to Question 1(e), where you evaluated the same integral using a substitution. Are your results compatible?

Integration by parts

6. Using integration by parts, determine the following integrals:

   (a) $\displaystyle\int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx$, (b) $\displaystyle\int_0^\infty xe^{-2x}\,dx$, (c) $\displaystyle\int_0^1 x\log\left(\frac{1}{x}\right)\,dx$, (d) $\displaystyle\int_0^\infty x^3e^{-x^2}\,dx$.

7. By writing each of the following functions $f(x)$ in the form $1 \cdot f(x)$, and using integration by parts, determine their indefinite integrals:

   (a) $\log(x)$, (b) $\log^3(x)$, (c) $\cosh^{-1}(x)$, (d) $\tanh^{-1}(x)$, (e) $\sin(\log(x))$.

Reduction formulae

8. (a) Show that for $n \geq 1$, we have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx + c,$$

   where c is an arbitrary constant. Hence, evaluate $\int \sin^6(x)\,dx$.

   (b) Using (a), show that the integral $I_n = \int_0^{\pi/2} \sin^n(x)\,dx$ satisfies $I_n = (n-1)I_n/n$. Hence, evaluate I_2 and I_4.

9. Establish reduction formulae for each of the following parametric integrals:

   (a) $I_n = \int_0^\infty x^n e^{-x^2}\,dx$, (b) $J_n = \int_0^\pi x^{2n}\cos(x)\,dx$, (c) $K_n = \int_0^\infty x^{n-1}e^{-x}\,dx$, (d) $L_n = \int_0^\infty \frac{dx}{(1+x^2)^n}\,dx$.

   Hence: (i) evaluate I_3, I_5; (ii) evaluate J_3, J_5; (iii) establish a general formula for K_n; (iv) evaluate L_4. $(*)$ Using part (c), suggest a reasonable definition of $z!$ where z is a complex number.

Miscellaneous integrals

[*This section contains a large collection of integrals from past papers for you to do. If you feel like you are getting too much of a good thing, feel free to save some of them for us to do together in the supervision.*]

10. Evaluate the following integrals, using the most efficient method in each case:

   (a) $\int_4^9 \frac{dx}{\sqrt{x}-1}$

   (b) $\int_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\,dx$

   (c) $\int \frac{e^{2x}-2e^x}{e^{2x}+1}\,dx$

   (d) $\int \frac{dx}{1+3\cos^2(x)}$

   (e) $\int_2^3 \frac{2x+1}{x(x+1)}\,dx$

   (f) $\int \frac{1}{2\sqrt{x}}e^{\sqrt{x}}\,dx$

   (g) $\int x^3 e^{-x^4}\,dx$

   (h) $\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)} + \frac{1}{x(\sin^2(x)+\log(x))}\right)dx$

   (i) $\int x\sqrt{3-2x}\,dx$

   (j) $\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx$

   (k) $\int \frac{\log(x)}{x^4}\,dx$

   (l) $\int \sqrt{1-x^2}\,dx$

   (m) $\int_{\pi/3}^{\pi/2} \tan(x)\cos^4(x)\,dx$

   (n) $\int_1^5 x^2\log(x)\,dx$

   (o) $\int e^x\sinh(3x)\,dx$

   (p) $\int \frac{\arctan(x)}{x^2}\,dx$

(q) $\displaystyle\int_{e^3}^{e^4} \frac{3\log(x) - 4}{x\log^2(x) - 3x\log(x) + 2x}\, dx$

(r) $\displaystyle\int_0^{\pi/6} x\sin(3x)\, dx$

(s) $\displaystyle\int \sin(2x)e^{\sin^2(x)}\, dx$

(t) $\displaystyle\int \frac{dx}{\cos^2(x)(\tan^3(x) - \tan(x))}$

(u) $\displaystyle\int_{-1/\pi}^{1/\pi} \sin^2(3x^3 + 2x)\log\left[\frac{1 - x^5}{1 + x^5}\right]\, dx$

(v) $\displaystyle\int \sin(2x)\cos(x)\, dx$

(w) $\displaystyle\int x\log(x)\, dx$

(x) $\displaystyle\int \frac{dx}{x\log(x)}$

(y) $\displaystyle\int \frac{\sinh^3(x)}{\cosh^2(x)}\, dx$

(z) $\displaystyle\int \frac{1}{\sin^2(3x+1)}\, dx$

The fundamental theorem of calculus

11. State both parts of the *fundamental theorem of calculus*. Use the fundamental theorem of calculus to simplify the following expressions:

$$\text{(a) } \frac{d}{dx}\int_1^x \frac{\log(t)\sin^2(t)}{t^2+7}\, dt, \quad \text{(b) } \frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2 + y^6)\, dy\right], \quad \text{(c) } \frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\, dt\right].$$

12. Without evaluating the integrals, determine the local extrema of the functions F_1, F_2 defined by:

$$\text{(a) } F_1(x) = \int_0^x t^2\sin^2(t)\, dt, \qquad \text{(b) } F_2(x) = \int_{-\infty}^x e^{-t^2}\, dt.$$

Hence, sketch the graphs of the functions F_1, F_2. [*Note:* $F_2(x) \to \sqrt{\pi}$ *as* $x \to \infty$; *this is because the value of* F_2 *approaches the* Gaussian integral, *which we shall study in Lent term*.]