Part IA: Mathematics for Natural Sciences B
Examples Sheet 1: Basics of vector geometry, and the scalar and vector products

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Basics of vector algebra

1. Let $A = (1, 3, 4)$, $B = (-1, 2, 4)$, and $C = (2, 2, 3)$. Which of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ is the longest?

2. (a) State the definition of $\mathbf{v} + \mathbf{w}$, given the vectors $\mathbf{v}, \mathbf{w}$. Using this definition, show that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$, and $(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$ for any vectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$.

   (b) Suppose that an aeroplane's engine produces a velocity 125 km h^{-1} due North. If there is a wind travelling at a velocity 80 km h^{-1} due North-West, use trigonometry to determine how fast the aeroplane travels across the Earth, and the bearing of its direction of travel from North.

3. (a) Define a *basis* of vectors.

   (b) Let $\mathbf{v} = (1, 2)$, $\mathbf{e}_1 = (1, -1)$ and $\mathbf{e}_2 = (2, 3)$. Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$, and determine the components of $\mathbf{v}$ with respect to this basis.

   (c) Let $\mathbf{w}_1 = (1, 2, 3)$ with respect to the basis $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$, and let $\mathbf{w}_2 = (3, 2, 1)$ with respect to the basis $\{(0, 1, 2), (2, 1, 0), (0, 1, -2)\}$. Find $\mathbf{w}_1 - 3\mathbf{w}_2$ with respect to the standard basis of $\mathbb{R}^3$.

The equation of a line

4. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be 3-vectors, and suppose that $\mathbf{w} \neq \mathbf{0}$.

   (a) Explain why the equation $\mathbf{r} = \mathbf{v} + \lambda\mathbf{w}$, as $\lambda \in \mathbb{R}$ varies, represents a line, and summarise its properties. Why is the condition $\mathbf{w} \neq \mathbf{0}$ necessary?

   (b) If $\mathbf{v} = (x_0, y_0, z_0)$ and $\mathbf{w} = (a, b, c)$, where $a, b, c \neq 0$, show that the same line may be equivalently described through the system of equations:
   $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
   What is the corresponding system of equations in the cases where one or more of a, b, c are zero?

   (c) Show that the position vectors $(1, 0, 1)$, $(1, 1, 0)$ and $(1, -3, 4)$ lie on a straight line, and find both its vector form, as in (a), and its Cartesian form, as in (b).

5. Show that the solution of the linear system $x + 2y + 3z = 0, 3x + 2y + z = 0$ is a line that is equally inclined to the x and z-axes, and makes an angle $\arccos(-\sqrt{2/3})$ with the y-axis.

6. (a) A *median* of a triangle is a line joining a vertex to the midpoint of its opposite edge. Prove that the three medians of a triangle are concurrent (the point at which they meet is called the *centroid* of the triangle).

   (b) Similarly, prove that in any tetrahedron, the lines joining the midpoints of opposite edges are concurrent.

The scalar product

7. Let $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ be 3-vectors.

   (a) Give the definition of the *scalar product* $\mathbf{v} \cdot \mathbf{w}$ in terms of lengths and angles. If $\mathbf{v}$ is a unit vector, explain why $\mathbf{v} \cdot \mathbf{w}$ is the signed length of the projection of $\mathbf{w}$ in the direction of $\mathbf{v}$.

(b) Using *only* this definition, prove each of the following properties of the scalar product:

(i) *commutativity*: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;

(ii) *homogeneity*: $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w})$;

(iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$.

[Hint: In (ii), consider the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ separately.]

Now let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.

(c) Using *only* the definition of the scalar product in terms of lengths and angles, show that for $i, j = 1, 2, 3$, we have $\mathbf{e}_i \cdot \mathbf{e}_j = 1$ if $i = j$, and $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$.

(d) By writing $\mathbf{v}$, $\mathbf{w}$ as linear combinations of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and applying the properties of the scalar product from the previous question, prove the formula $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.

8. Explain how we can use the two different formulae for the scalar product to determine the angles between vectors. Hence:

   (a) determine the angles AOB and OAB, where the points A, B have coordinates $(0, 3, 4)$, $(3, 2, 1)$ respectively;

   (b) find the acute angle at which two diagonals of a cube intersect.

9. Consider the line with vector equation $\mathbf{r} = (1, 0, 1) + \lambda(3, 2, 1)$, where λ is a real parameter.

   (a) Using the scalar product, compute the projection of the vector $(1, 2, 3)$ in the direction $(3, 2, 1)$.

   (b) Hence, determine the point on the line which is closest to the point $(0, 2, 4)$, and the shortest distance from the line to the point $(0, 2, 4)$.

   (c) Now, generalise your result: find a formula for the point on the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ which is closest to the point with position vector $\mathbf{p}$, and a formula for the the shortest distance from the line to the point.

10. Show that if four points A, B, C, D are such that $AD \perp BC$ and $BD \perp AC$, then $CD \perp AB$.

11. Using the scalar product, prove that for any tetrahedron, the sum of the squares of the lengths of the edges equals four times the sum of the squares of the lengths of the lines joining the mid-points of opposite edges.

12. (a) Using the geometric definition of the scalar product, prove the *Cauchy-Schwarz inequality* $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

    (b) From the Cauchy-Schwarz inequality, deduce the *triangle inequality* $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. What is the geometrical significance of this inequality? Learn this equality off by heart; it will be useful later when we study limits!

    (c) From the triangle inequality, deduce the *reverse triangle inequality* $||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} - \mathbf{b}|$.

The equation of a plane

13. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be fixed 3-vectors, with $\mathbf{b} \neq \mathbf{0}$.

    (a) Explain why the equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ represents a plane, and summarise its properties. Show using properties of the scalar product that an equivalent representation of this plane is $\mathbf{r} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. What is the geometric significance of the quantity $|\mathbf{a} \cdot \mathbf{b}|/|\mathbf{b}|$ here?

    (b) By writing $\mathbf{r} = (x, y, z)$, $\mathbf{b} = (l, m, n)$, and $\mathbf{a} \cdot \mathbf{b} = d$, show that the equation of a plane may equivalently be written in the Cartesian form $lx + my + nz = d$.

    (c) Find the equation of the plane containing the point $(3, 2, 1)$ with normal $(1, 2, 3)$ in both the vector form, as in (a), and the Cartesian form, as in (b). What is the shortest distance from the origin to the plane?

14. Consider the plane with vector equation $(\mathbf{r} - (1,0,1)) \cdot (2,-1,0) = 0$.
    (a) Using the scalar product, compute the projection of the vector $(2,0,3)$ in the direction $(2,-1,0)$. Hence, determine the length of the projection of the vector $(2,0,3)$ in the plane. [*Hint: Pythagoras.*]
    (b) Using the results of part (a), determine the point on the plane which is closest to the point $(3,0,4)$, and the shortest distance from the plane to the point $(3,0,4)$.
    (c) Now, generalise your result: find a formula for the point on the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ which is the closest to the point with position vector $\mathbf{p}$, and a formula for the shortest distance from the plane to this point.
15. Using the results of Question 15, calculate the shortest distances between the plane $5x + 2y - 7z + 9 = 0$ and the points $(1,-1,3)$ and $(3,2,3)$. Are the points on the same side of the plane?

Equations of other 3D surfaces

16. Let k, m be positive constants, with $m < 1$. Describe the following surfaces: (a) $|\mathbf{r}| = k$; (b) $\mathbf{r} \cdot \mathbf{u} = m|\mathbf{r}|$.
17. Describe the surface given by the vector equation:
$$|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = 2,$$
where $\mathbf{u} = \frac{1}{\sqrt{2}}(1,0,1)$. What is the intersection of this surface and the surface $x + z = 0$?
18. (a) Write down a vector equation for the sphere with centre at the point with position vector $\mathbf{a}$, and radius $p > 0$.
    (b) If there is a second sphere with centre at the point with position vector $\mathbf{b}$, and radius $q > 0$, what conditions are required on $\mathbf{a}$, $\mathbf{b}$, p and q for the two spheres to intersect in a circle?
    (c) Show that, if the two spheres do intersect, then the plane in which their intersection occurs is given by the equation $2\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) = p^2 - q^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2$.

The vector product

19. Let $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ be 3-vectors.
    (a) Give the geometrical definition of the *vector product* (or *cross product*) $\mathbf{v} \times \mathbf{w}$ in terms of lengths, angles and an appropriate perpendicular vector.
    (b) Using *only* this definition, prove each of the following properties of the vector product:
        (i) *anti-commutativity*: $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$;
        (ii) *homogeneity*: $(\lambda \mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w})$;
        (iii) *left-distributivity*: $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$.

        [Hint: In (ii), consider the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ separately. In (iii), start by explaining why $\mathbf{v} \times \hat{\mathbf{u}}$ is the projection of $\mathbf{v}$ onto the plane through the origin perpendicular to $\mathbf{u}$, followed by a rotation by $\frac{1}{2}\pi$.]

    Now let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ be the standard basis vectors for $\mathbb{R}^3$, and let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be 3-vectors.

    (c) Using *only* the definition of the vector product in terms of lengths, angles, and a perpendicular vector, show that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.
    (d) By writing $\mathbf{v}, \mathbf{w}$ as linear combinations of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and applying the properties of the vector product from the previous question, prove the standard formula:
$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, \quad v_3 w_1 - v_1 w_3, \quad v_1 w_2 - v_2 w_1).$$
20. Find the angle between the position vectors of the points $(2,1,1)$ and $(3,-1,-5)$, and find the direction cosines of a vector perpendicular to both. Can both the angle and vector be computed using *only* the vector product?
21. Find all points $\mathbf{r}$ which satisfy $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (1,1,0)$ and $\mathbf{b} = (1,-1,0)$.
22. Using properties of the vector product, prove the identity $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$.