Part IA: Mathematics for Natural Sciences B
Examples Sheet 10: First-order ordinary differential equations

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions.

Separable equations

1. Explain what is meant by a *separable differential equation*, and how we can solve one. Find the general solution of the following separable differential equations:

   (a) $\dfrac{dy}{dx} = x$, (b) $\dfrac{dy}{dx} = (2-y)(1-y)$, (c) $\dfrac{dy}{dx} = -\dfrac{x^3}{(1+y)^2}$,

   (d) $\dfrac{dy}{dx} = \dfrac{4y}{x(y-3)}$, (e) $\dfrac{dy}{dx} = xe^{x-2y}$, (f) $\dfrac{dy}{dx} = \sin(y+x) - \sin(y-x)$.

2. Find the solution of the differential equation:
$$\frac{dy}{dx} = 1 - y^2,$$
   subject to the initial condition $y(0) = y_0$. Sketch representative solution curves for various values of y_0. Could you have drawn the sketches without solving the equation?

3. Determine the half-life of thorium-234 if a sample of mass 5g is reduced to 4g in one week. What amount of thorium is left after twelve weeks?

4. *Newton's law of cooling* states that the rate of heat loss from a body is proportional to the difference between the temperature of the object and its ambient environment. Assuming that Newton's law of cooling applies, calculate the time at which a cup of tea in a $20°$C room was made, given that: (i) the tea is measured to have temperature $40°$C at 5pm; (ii) the tea is measured to have temperature $30°$C at 3pm; (iii) the water was initially at boiling point.

5. Consider the family of curves $C = \{y = ax^2 : a \in \mathbb{R}\}$. Sketch a few representative curves in C. Determine a family of curves $C' = \{y = f(x,b) : b \in \mathbb{R}\}$ such that each curve in C' is orthogonal to all curves in C, and sketch a few representative curves in the family C'.

6. ($*$) Show by separating variables that a valid solution of the equation:
$$\frac{dy}{dt} = 2\sqrt{y},$$
   satisfying the initial condition $y(0) = 0$, is:
$$y(t) = \begin{cases} 0, & 0 \le t \le c, \\ t^2, & c \le t, \end{cases}$$
   for any constant c. Comment on this result in relation to the uniqueness of solutions of differential equations. If you are studying IA Physics, what does this result tell you about the deterministic nature of Newtonian mechanics? [*Hint: Suppose that y is a velocity.*]

Linear equations

7. Write down the general form of a *linear first-order differential equation*. What is meant by an *integrating factor* for such an equation? Solve the following linear first-order differential equations by finding an appropriate integrating factor:

(a) $\frac{dy}{dx} + 2xy = 4x,$ (b) $\frac{dy}{dx} + \frac{y}{(2-3x^2)^3} = 1,$ (c) $\frac{dy}{dx} - y\tan(x) = 1,$

(d) $\frac{dy}{dx} + (1+\log(x))y = x^{-x},$ (e) $\frac{dy}{dx} + \frac{y}{\tan(x)} = \cos^2(x),$ (f) $\frac{1}{x}\frac{dy}{dx} + y - 5e^{-x^2} = 0.$

8. Solve the equation:

$$(x^2+1)\frac{dy}{dx} + 3x^3 y = 6xe^{-3x^2/2},$$

subject to the boundary condition $y(0) = 1$.

9. Establish a formula for the general solution of the linear differential equation:

$$\alpha(x)\frac{dy}{dx} + \beta(x)y = \gamma(x),$$

stating any conditions you must assume for your formula to be valid.

Some 'easy to spot' substitutions

[*Hint: most 'easy' substitutions come from thinking about implicit differentiation*.]

10. Using an appropriate substitution, find the general solution of the equation:

$$y^3 + x + 3y^2\frac{dy}{dx} = 0.$$

Find also the solution satisfying the boundary condition $y(0) = 1$.

11. Using an appropriate substitution, find the general solution of the equation:

$$(x+y+1)^2\frac{dy}{dx} + (x+y+1)^2 + x^3 = 0.$$

Find also the solution satisfying the boundary condition $y(0) = 0$.

Some standard substitutions: homogeneous, Bernoulli, and affine transformations

12. Define a *homogeneous equation*, and state the substitution which renders them solvable. Hence, solve the equations:

(i) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right),$ (ii) $(y-x)\frac{dy}{dx} + (2x+3y) = 0,$ (iii) $\frac{dy}{dx} = \frac{x^3+y^2}{3xy^2}.$

13. Define a *Bernoulli equation*, and state the substitution which renders them solvable. Hence, solve the equations:

(i) $\frac{dy}{dx} - y = xy^5,$ (ii) $\frac{dy}{dx} + y = y^2(\cos(x) - \sin(x)),$ (iii) $xy\frac{dy}{dx} + (x^2+y^2+x) = 0.$

14. (a) Show that equations of the form:

$$\frac{dy}{dx} = f(ax+by+c),$$

with $b \neq 0$ may be reduced to a separable equation by making the substitution $u = ax+by+c$.

(b) Hence, solve the equations:

(i) $\frac{dy}{dx} = (4x+y)^3,$ (ii) $\cos(x+y-1)\frac{dy}{dx} = \sin(x+y-1),$ (iii) $\frac{dx}{dy} = \operatorname{sech}^2(2x-y+2).$