Part IA: Mathematics for Natural Sciences B
Examples Sheet 11: Linear ordinary differential equations

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with a $(*)$ are difficult and should not be attempted at the expense of the other questions.

Basic definitions

1. Consider the general linear nth-order ordinary differential equation:

$$\alpha_n(x)\frac{d^n y}{dx^n} + \alpha_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \alpha_1(x)\frac{dy}{dx} + \alpha_0(x)y = f(x).$$

where $\alpha_n(x) \not\equiv 0$.

(a) Give the definitions of the following terms: (i) homogeneous equation; (ii) coefficient functions; (iii) forcing.

(b) Define a *complementary function* for this equation. How many arbitrary constants feature in the complementary function for this equation?

(c) Define a *particular integral* for this equation. Is a particular integral for this equation unique?

(d) Show that if y_{CF} is the complementary function for this equation, and y_{PI} is a particular integral, then the sum $y = y_{\text{CF}} + y_{\text{PI}}$ solves the equation.

(e) Suppose that we now seek a particular solution of this equation satisfying certain boundary conditions. How many boundary conditions are needed to fully specify a particular solution?

2. By direct differentiation, verify that the following ordinary differential equations have the given complementary functions:

(a) $y_{\text{CF}} = Ax + Be^x$ is the complementary function for $(x-1)y'' - xy' + y = 0$;

(b) $y_{\text{CF}} = A + B\log(x)$ is the complementary function for $xy'' + y' = \cos(x)e^{x^2}$;

(c) $y_{\text{CF}} = Ax + B\sin(x)$ is the complementary function for $(1 - x\cot(x))y'' - xy' + y = x$;

(d) $y_{\text{CF}} = A + Bx + Ce^x$ is the complementary function for $y''' - y'' = x$.

3. By direct differentiation, verify that the following ordinary differential equations have the given particular integrals:

(a) $y_{\text{PI}} = \cos(x)$ is a particular integral for $-y'' + y = 2\cos(x)$;

(b) $y_{\text{PI}} = x^2$ is a particular integral for $xy'' + y' = 4x$;

(c) $y_{\text{PI}} = e^{x^2}$ is a particular integral for $y''' - 2xy'' - 2y' - y = (4x-1)e^{x^2}$;

(d) $y_{\text{PI}} = \sin(x)/x$ is a particular integral for $xy^{(4)} + 4y^{(3)} + xy^{(2)} + 2y^{(1)} + xy = \sin(x)$.

4. Verify that the equation:

$$(3+x)y'' + (2+x)y' - y = x^2 + 6x + 6$$

has complementary function $y_{\text{CF}}(x) = Ae^{-x} + B(x+2)$. Hence, by finding a particular integral of the form

$$y_{\text{PI}}(x) = \alpha x^2 + \beta x + \gamma,$$

determine the full solution to the equation subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$.

Constant coefficient equations

5. Consider the linear second-order ordinary differential equation with *constant coefficients*:

$$\alpha \frac{d^2 y}{dx^2} + \beta \frac{dy}{dx} + \gamma y = f(x),$$

where α, β, γ are *constants*, with $\alpha \neq 0$.

(a) Show that the equation may be rewritten in the 'factorised' form:

$$\alpha \left(\frac{d}{dx} - \omega_1 \right) \left(\frac{d}{dx} - \omega_2 \right) y = f(x),$$

where ω_1, ω_2 are the roots of the *auxiliary equation* $\alpha\mu^2 + \beta\mu + \gamma = 0$.

(b) Deduce that the complementary function of this equation is:

$$y_{\mathsf{CF}}(x) = \begin{cases} Ae^{\omega_1 x} + Be^{\omega_2 x}, & \text{if } \omega_1 \neq \omega_2, \\ (A + Bx)e^{\omega x}, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

How does this result generalise to an nth order differential equation of this form?

(c) $(*)$ Deduce also that we may construct an analytic particular integral, given by:

$$y_{\mathsf{PI}}(x) = \frac{1}{\alpha} e^{\omega_2 x} \int_{x_0}^{x} \left(e^{(\omega_1 - \omega_2)\eta} \int_{\eta_0}^{\eta} e^{-\omega_1 \xi} f(\xi) d\xi \right) d\eta,$$

where x_0, η_0 are arbitrary constants. By setting $\eta_0 = x_0$ and changing the order of integration in the double integral, deduce the simpler form:

$$y_{\mathsf{PI}}(x) = \begin{cases} \dfrac{1}{\alpha(\omega_1 - \omega_2)} \displaystyle\int_{x_0}^{x} \left(e^{\omega_1 (x - \xi)} - e^{\omega_2 (x - \xi)} \right) f(\xi) \, d\xi, & \text{if } \omega_1 \neq \omega_2, \\ \dfrac{1}{\alpha} \displaystyle\int_{x_0}^{x} (x - \xi) e^{\omega (x - \xi)} f(\xi) \, d\xi, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

[In practice, it is often just easier to guess a particular integral rather than use this formula, though!]

6. Determine the solutions of the following differential equations:

(a) $y'' + 6y' + 5y = 0$;

(b) $y'' + 3y' + 4y = 0$;

(c) $y'' + 4y = x$;

(d) $y'' - 2y' + 2y = 2x^2$;

(e) $y'' + y = |x|$;

(f) $y'' + 3y' + 2y = e^{-x}$;

(g) $y'' - 2y' + 5y = e^x \cos(2x)$;

(h) $y'' + 2y' + y = 2xe^{-x}$.

7. Determine the solutions of the following differential equations subject to the given constraints:

(a) $y'' - 4y' + 13y = 0$, subject to $y(0) = \pi$ and $y(-\pi/2) = 1$;

(b) $y'' - 4y' + 5y = 125x^2$, subject to $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = \frac{25\pi^2}{4} + 20\pi + 22$;

(c) $y'' + 7y' + 12y = 6$, subject to $y(0) = 0$ and $y\left(\frac{1}{3}\right) = \frac{1 - e^{-1}}{2}$;

(d) $y'' + 7y' + 12y = 2e^{-3x}$, subject to $y(0) = 1$ and $y'(0) = 0$.

8. Find the value of a for which the complementary function of the ODE:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + ay = 0,$$

is given by $y_{\text{CF}} = Axe^{-2x} + Be^{-2x}$.

9. Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + y = \cos(kx),$$

where k is a real number.

10. The differential operator $\mathcal{L}$ is defined by:

$$\mathcal{L} = \frac{d^2}{dx^2} + \sqrt{3}\frac{d}{dx} + 3.$$

Solve the equation $\mathcal{L}y = 0$, and hence solve the equations:

(a) $\mathcal{L}y = e^{-\sqrt{3}x}$;

(b) $\mathcal{L}y = x$.

Without further calculation, state the general solution of $\mathcal{L}y = 2x + e^{-\sqrt{3}x}$. Find also the solution of this equation satisfying the boundary conditions:

$$y(0) = 0, \qquad y(\pi) = \frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}}.$$

Harmonic oscillators

11. Consider the constant coefficient linear second-order ordinary differential equation:

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + \omega_0^2 y = f(t),$$

modelling an oscillating system which depends on time t. The coefficients γ, ω_0 are positive.

(a) What is the physical interpretation of the constant γ? What is the physical interpretation of the function $f(t)$?

(b) Find the complementary function of this equation. Discuss the different forms the complementary function can take (in particular, defining the terms *underdamping*, *critical damping*, and *overdamping*), and how this relates to the *transient* behaviour of the oscillator.

(c) In the underdamped case, find the long-term behaviour of the oscillator in the case of resonant forcing:

$$f(t) = e^{-\gamma t}\sin\left(t\sqrt{\omega_0^2 - \gamma^2}\right).$$

Coupled systems of differential equations

[*This section is labelled 'non-examinable' in the lecture notes, but has appeared on Tripos papers - see e.g. 2023 Paper 2 or 2021 Paper 2.*]

12. (a) Consider the system of differential equations:

$$\frac{dx}{dt} = ax + by + p, \qquad \frac{dy}{dt} = cx + dy + q,$$

for the variables $x(t), y(t)$, where a, b, c, d, p, q are constants. Show that:

$$\frac{d^2x}{dt^2} = (a+d)\frac{dx}{dt} + (bc - ad)x + bq - pd.$$

(b) Hence:

(i) Find the general solution of the system:

$$\frac{dx}{dt} = 4y + 2, \qquad \frac{dy}{dt} = x.$$

(ii) Solve the system:

$$\frac{dx}{dt} = 3x - y, \qquad \frac{dy}{dt} = x + y,$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 1$.

(iii) Solve the system:

$$\frac{dx}{dt} = -3x + y, \qquad \frac{dy}{dt} = -5x + y,$$

subject to the initial conditions $x(0) = 1, y(0) = 1$.

($*$) Equidimensional equations

[*This section is not lectured, but is very useful if you choose to do Part IB Mathematics in second-year.*]

13. Consider a linear second-order ordinary differential equation with *non-constant coefficients*:

$$\alpha x^2 \frac{d^2y}{dx^2} + \beta x \frac{dy}{dx} + \gamma y = f(x),$$

where α, β, γ are *constants*, with $\alpha \neq 0$. This type of equation is called an *equidimensional equation*. If you are doing Part IA Physics, suggest a reason for this name.

(a) Show that the equation may be written in the form:

$$\alpha\left(x\frac{d}{dx} - \omega_1\right)\left(x\frac{d}{dx} - \omega_2\right) = f(x),$$

where ω_1, ω_2 are the roots of the *auxiliary equation* $\alpha\mu(\mu - 1) + \beta\mu + \gamma = 0$.

(b) Deduce that the complementary function of this equation is:

$$y_{\text{CF}}(x) = \begin{cases} Ax^{\omega_1} + Bx^{\omega_2}, & \text{if } \omega_1 \neq \omega_2, \\ (A + B\log(x))x^{\omega}, & \text{if } \omega_1 = \omega_2 = \omega. \end{cases}$$

How does this result generalise to an nth order differential equation of this form?

14. Using the results of Question 13, determine the solutions of the following differential equations:

(a) $x^2y'' - 2xy' + y = 0$, subject to the initial data $y(1) = 1, y'(1) = 0$;

(b) $x^2y'' - xy' + y = x^2$, subject to the initial data $y(1) = 2, y'(1) = 3$;

(c) $x^2y'' - xy' + y = x\log(x)$, subject to the initial data $y(1) = 0, y'(1) = 1$.