Part IA: Mathematics for Natural Sciences B
Examples Sheet 12: Partial differentiation, differentials, and the single-variable chain rule with multivariable functions

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Questions marked with a $(*)$ are difficult and should not be attempted at the expense of the other questions.

Partial differentiation: definitions and basic examples

1. Let $f \equiv f(x,y)$ be a function of x and y.
   (a) Define the *partial derivatives* $\partial f/\partial x$ and $\partial f/\partial y$ in terms of limits. Define also the gradient $\nabla f(x,y)$.
   (b) Determine the gradient of the following functions:
   (i) $f = x^3 - 2x^2y + 3xy^3 - 4y^3$, (ii) $f = \exp(-x^2y^2)$, (iii) $f = \exp(-x/y)$, (iv) $f = \sin(x+y)$.
   (c) For each of the functions in part (b), compute the four possible second partial derivatives. Verify that in each case we have symmetry of the mixed partial derivatives.

2. Show that the function:
$$w(x,y) = \frac{1}{360}\left(15x^4y^2 - x^6 + 15x^2y^4 - y^6\right)$$
is a solution of the equation:
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = x^2y^2.$$

3. Show that the function:
$$\phi(x,t) = \frac{1}{\sqrt{4\pi\sigma^2 t}}\exp\left(-\frac{(x-x_0)^2}{4\sigma^2 t}\right),$$
where $t > 0, x_0, \sigma$ are real positive constants, and $\sigma^2 \neq 0$, is a solution of the equation:
$$\frac{\partial \phi}{\partial t} = \sigma^2 \frac{\partial^2 \phi}{\partial x^2}.$$

4. $(*)$ Show that the mixed partial derivatives of the function:
$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{for } (x,y) \neq (0,0), \\ 0, & \text{for } (x,y) = (0,0), \end{cases}$$
are not symmetric at the point $(0,0)$. Why is this allowed to occur here?

Integration and basic partial differential equations

5. Let $f \equiv f(x,y)$ be a function of x and y. Find the general solution of the following partial differential equations:
$$\text{(a) } \frac{\partial f}{\partial x} = xy^2 + \cos(x), \qquad \text{(b) } \frac{\partial f}{\partial y} = y^2 - xe^y, \qquad \text{(c) } \frac{\partial^2 f}{\partial x^2} + y^2 f = x, \qquad \text{(d) } \frac{\partial^2 f}{\partial x \partial y} = 0.$$

6. Let $f \equiv f(x,y)$ be a function of x and y. Find the solution of the following partial differential equations, subject to the given boundary conditions:
$$\text{(a) } \frac{\partial f}{\partial x} = xy^2, \text{ where } f(0,y) = y^3, \qquad \text{(b) } y^3\frac{\partial f}{\partial y} = x, \text{ where } \lim_{y\to\infty} f(x,y) = e^x.$$

Differentials

7. In lectures, differentials are introduced as 'infinitesimal quantities'; however, there is no need for this, and the concept can easily be made mathematically precise. In real multivariable calculus, we can view the differential as alternative notation for the gradient, $df \equiv \nabla f$.[1]

   (a) Using this definition, show that $d(x^2 + y^2) = 2xdx + 2ydy$ and $d(x^2y) = 2xydx + x^2dy$.

   (b) Generalising your argument, show that for any smooth function $f(x, y)$, we have:

   $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

   as stated in lectures.

8. By computing the partial derivatives, determine the differentials of each of the following functions in terms of the differentials of x and y:

   (a) $\exp(-1/(x + y))$, (b) $\sinh(x)/\sinh(y)$, (c) $\sqrt{x^2 + y^2}$, (d) $\arctan(y/x)$, (e) x^y.

9. Let f, g be functions of (x, y), let a, b be constants, and let $F : \mathbb{R} \to \mathbb{R}$ be any differentiable single-variable function. Prove the following basic properties of differentials:

   (a) $d(af + bg) = adf + bdg$, (b) $d(fg) = fdg + gdf$, (c) $d(F(f)) = F'(f)df$.

   Hence, without computing partial derivatives, show that if $f(x, y) = \log(xy^2)$, we have:

   $$df = \frac{dx}{x} + \frac{2\,dy}{y}.$$

   Now, verify that your result is correct by computing the partial derivatives of $f(x, y)$.

10. The period T of a simple pendulum can be approximated by the formula:

    $$T = 2\pi\sqrt{\frac{l}{g}},$$

    where l is the length of the pendulum, and g is gravitational acceleration.

    (a) By taking logarithms, show that:

    $$\frac{dT}{T} = \frac{dl}{2l} - \frac{dg}{2g}.$$

    (b) Hence, estimate the percentage change in the period of a pendulum if: (i) the length is increased by 0.1%; (ii) gravitational acceleration increased by 0.2%.

11. The magnitude of the gravitational force between two points masses m_1, m_2 which are separated by a distance $r > 0$ in three dimensional space is given by:

    $$F(r, m_1, m_2) = \frac{Gm_1m_2}{r^2},$$

    where G is a positive constant. Find dF in terms of dr, dm_1 and dm_2. Hence compute the (approximate) fractional change in distance if there is no change in the force, and the masses of both particles increase by 1%.

[1] It might seem a bit silly to have two notations for the same thing, the differential and the gradient. However, the concepts become different when you start doing calculus on *curved* spaces instead of the 'flat' space $\mathbb{R}^2$. You will see this in the Relativity course in Part II Physics, should you take it.

12. The energy, $E(m, v)$, of a relativistic particle of rest mass m and speed v is given by:

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}},$$

where c, the speed of light, is a constant.

(a) Find dE in terms of dm, dv.

(b) Two particles, A, B, have equal energy and move at 90% and 91% of the speed of light respectively. Particle A has rest mass m_A. What is the (approximate) difference in the rest masses of the particles, in terms of m_A? Which particle has the larger rest mass?

13. The differential of the volume V of a geometrical figure is given by:

$$dV = 2\pi rhdr + \pi r^2 dh,$$

where r and h are non-negative parameters and the volume vanishes when these parameters are zero. Find an expression for the fractional change in volume dV/V for fractional changes in the parameters dr/r and dh/h. Find dV/V if r increases by 1% and h increases by 2%.

Multivariable Taylor series, and error propagation

14. Find, up to and including terms of quadratic order, the Taylor series of the functions:

(a) $f(x, y) = \sin(x + 2y)$ about the point $(x, y) = (\pi/2, 0)$;

(b) $f(x, y) = e^x \cos(y)$ about the point $(x, y) = (0, 0)$.

15. Let $f(X, Y)$ be a function of the independent random variables X and Y, and let $\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$. Using properties of variance, and multivariable Taylor series, show that:

$$\mathrm{Var}(f(X,Y)) \approx \left(\frac{\partial f}{\partial X}\right)^2 \mathrm{Var}(X) + \left(\frac{\partial f}{\partial Y}\right)^2 \mathrm{Var}(Y),$$

where the partial derivatives are evaluated at the mean $(X, Y) = (\mu_X, \mu_Y)$. Deduce the standard formula for error propagation ('adding errors in quadrature'):

$$\Delta f(X,Y) = \sqrt{\left(\frac{\partial f}{\partial X}\right)^2 (\Delta X)^2 + \left(\frac{\partial f}{\partial Y}\right)^2 (\Delta Y)^2}.$$

($*$) If you are taking Part IA Physics, check that this agrees with the results stated therein when $f(X, Y) = X + Y$ and $f(X, Y) = X/Y$.

16. ($*$) If you are taking Part IA Physics, use the formula for propagation of error to determine the error in gravitational acceleration as determined from the period of a simple pendulum when the relative error in the string length is 0.1% and the relative error in the period is 0.2%.

The single-variable chain rule, with multivariable functions

17. Let $z(x, y)$ be a function defined implicitly by the equation:

$$x - \alpha z = \phi(y - \beta z),$$

where α, β are real constants, and ϕ is an arbitrary differentiable function. Show that z satisfies the partial differential equation:

$$\alpha \frac{\partial z}{\partial x} + \beta \frac{\partial z}{\partial y} = 1.$$

[*Hint: you can still use the normal single-variable chain rule here when taking each of the partial derivatives! Why?*]

18. Consider the function $u(x, y) = x\phi(y/x)$, where ϕ is a differentiable function of its argument and $x \neq 0$. Show that u satisfies:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u.$$

19. If $u(x, y) = \phi(xy) + \sqrt{xy}\psi(y/x)$, where ϕ and ψ are twice-differentiable functions of their arguments, show that u satisfies the partial differential equation:

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

20. Consider the partial differential equation:

$$2y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = xy(2y^2 - x^2).$$

   (a) Show that $u(x, y) = \phi(x^2 + 2y^2)$ is a solution of the homogeneous version of this equation, where ϕ is an arbitrary differentiable function.

   (b) By considering $u_p(x, y) = Ax^m y^n$ for some constants A, m, n, find a particular integral for this equation.

   (c) Hence, find the complete solution of the equation subject to the boundary condition $u(x, 1) = x^2$.

21. Consider the partial differential equation:

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2},$$

where $\lambda > 0$.

   (a) Show that $u(x, t) = (t + a)^{-1/2} v(y)$, where $y = (t + a)^{-1/2}(x + b)$, solves the equation (for appropriate constants a, b, which you should determine) if and only if v satisfies the ordinary differential equation:

$$-\frac{1}{2}\left(v + y\frac{dv}{dy}\right) = \lambda \frac{d^2 v}{dy^2}. \qquad (*)$$

   (b) Verify that $(*)$ has a solution of the form $v(y) = e^{-cy^2}$ for appropriately chosen c.

   (c) Using parts (a) and (b), find the solution of the original partial differential equation subject to the boundary condition:

$$u(x, 0) = \exp\left(-(x + 1)^2\right) + \exp\left(-(x - 1)^2\right).$$