Part IA: Mathematics for Natural Sciences B
Examples Sheet 14: Contour sketching, and constrained optimisation

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Questions marked with a $(*)$ are difficult and should not be attempted at the expense of the other questions. A section marked with a (†) contains content that is unique to the Mathematics B course.

Classification of stationary points

1. Describe the classification of stationary points of a function of two variables $f(x, y)$ via conditions on the second partial derivatives of the function at the stationary point. Hence or otherwise, sketch the contours of the functions:

   (a) $f(x, y) = x^2 - 4xy + y^2$;

   (b) $f(x, y) = x^2 + xy + y^2$;

   (c) $f(x, y) = x^2 + 2xy + y^2$.

   Add some arrows to your sketches, indicating the direction of the gradient vector at salient points.

Contour sketching

[*This section is supposed to give you some ideas for things you might consider when sketching the contours of a function, outside of just classifying the stationary points using the second derivative test. Often only the second derivative test is lectured!*]

2. (**Large x, y behaviour**) Consider the function:

$$f(x, y) = x^2 + xy + y^2 - x^3.$$

   (a) Show that this function has two stationary points, and classify them using the second derivative test. Find also the equations of the contour lines passing through any saddles, if there are any.

   (b) Explain why, for very large x, y, the contours become roughly vertical. [*The large x, y behaviour is often a useful point to consider when preparing a contour plot.*]

   (c) Hence, sketch the contours of the function.

3. (**Symmetry**) Consider the function:

$$f(x, y) = x^2 + 4xy + y^2 + x^4 + y^4.$$

   (a) Show that this function has two stationary points, and classify them using the second derivative test. Find also the equations of the contour lines passing through any saddles, if there are any.

   (b) Explain why, for very large x, y, the contours begin to resemble $x^4 + y^4 \approx k$ for k constant. What does this figure look like?

   (c) Explain why this function is symmetrical in the lines $y = x$ and $y = -x$. [*Symmetry in the lines $x = 0, y = 0$ or the lines $y = x, y = -x$ is often a useful point to consider when preparing a contour plot.*]

   (d) Hence, sketch the contours of the function.

4. (**Zero contours**) Consider the function:
$$f(x,y) = \frac{x+y}{x^2+y^2+1}.$$
   (a) Show that this function has two stationary points.
   (b) Find the equations of the zero contour(s) of the function. Explain also why the function tends to zero for very large x, y. Hence, classify the stationary points you found in (a), and thus sketch the contours of the function. [*If you can calculate them, the zero contours of a function are often a useful tool in classifying stationary points* **without** *using the - rather cumbersome - second derivative test.*]
   (c) Now, find the analytical equations of the contours of the function, and hence verify that the sketch you produced in part (b) is accurate. [*If you can calculate them, the analytical equations of the contours can be a very useful check!*]

5. (**Failure of the second derivative test**) Consider the function:
$$f(x,y) = x^2y^2 + x^4 + y^6.$$
   (a) Show that this function has a single stationary point, but that the second derivative test fails to classify it.
   (b) Using another method, classify it, and sketch the contours of the function.

6. (**Reduction to one dimension, and polar coordinates**) Consider the function:
$$f(x,y) = x^4 + y^4 - x^2y^2.$$
   (a) Show that this function has a single stationary point, but that the second derivative test fails to classify it.
   (b) By examining the behaviour of the function along the lines $y = x$, $y = -x$, $x = 0$ and $y = 0$, classify the stationary point, and hence sketch the contours of the function. [*Examining a function along specific lines through a stationary point reduces the problem to a one-dimensional problem, often aiding in classification of a point.*]
   (c) Show that the function can be rewritten in terms of polar coordinates as:
$$f(r,\theta) = \frac{1}{8}r^4\left(3\cos(4\theta)+5\right).$$
   Hence, verify that the sketch you provided in part (b) is accurate. [*Sometimes, switching into polar coordinates can be very helpful in understanding the nature of a stationary point - particularly in cases where the function factorises as $g(r)h(\theta)$ in polar form.*]

7. (**Another polar example**) Consider the function:
$$f(x,y) = (xy-y)e^{2x-x^2-y^2}.$$
   By writing $x - 1 = r\cos(\theta), y = r\sin(\theta)$, express the function in polar coordinates about $(1,0)$. Hence show that the function has five stationary points, classify them, and produce the contour plot for the function. How does this analysis compare with the second derivative approach?

Miscellaneous contour sketching

[*This section contains a large number of functions of two variables for which you are asked to produce contour plots. If you are getting bored, feel free to skip some and leave them for the supervision.*]

8. In light of Questions 2-7, make a list of things you could consider when producing a contour plot of a function of two variables $f(x,y)$.

9. Produce contour plots for each of the following functions, in each case: (i) indicating the nature and location of the stationary points; (ii) providing equations for the contour lines through saddles; and (iii) adding some arrows to your sketch showing the direction of the gradient vector ∇f at different points.

   (a) $x^4 + y^4 - 36xy$;

   (b) $4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4$;

   (c) $2x^3 + 6xy^2 - 3y^3 - 150x$;

   (d) $\frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$;

   (e) $xy(x^2 + y^2 - 1)$;

   (f) $x^6 + y^6 - 24x^2y^2$;

   (g) $x\ln(x) - y^2$;

   (h) $(1 - x^2 - y^2)^2 + \frac{7}{2}\ln(3 + y^2)$;

   (g) $\dfrac{1}{x^2 + y^2 + 1}$;

   (h) $\dfrac{2(y - x^2)}{1 + x^2 + y^2}$;

   (i) $(x^2 - y^2)e^{-x^2-y^2}$;

   (j) $(1 - 4xy)\exp(-x^2 - y^2)$;

   (k) $e^x + e^y - x - y$;

   (l) $e^x - e^y + x - y$;

   (m) $\exp(\frac{1}{2}(x^2 + y^2) + \frac{1}{4}x^4)$;

   (n) $\exp(\frac{1}{2}(x^2 + y^2) - \frac{1}{4}x^4)$;

   (o) $\sin(x)\sin(y)$;

   (p) $1 - \cos(x) + \frac{1}{2}y^2$.

(†) Constrained optimisation using Lagrange multipliers

10. Explain the rationale behind the method of Lagrange multipliers for extremising a function $f(x, y)$ subject to a constraint $g(x, y) = 0$. How does the method of Lagrange multipliers generalise to extremising a function $f(x_1, ..., x_n)$ subject to multiple constraints $g_1(x_1, ..., x_n) = 0$, $g_2(x_1, ..., x_n) = 0$, etc?

11. Find the extremal values of the functions: (a) $f_1(x, y) = xy^2$; (b) $f_2(x, y) = e^{-xy}$, subject to the constraint that $x^2 + y^2 = 1$. Now, verify that your answers are correct by solving the constraint by setting $x = \cos(\theta)$, $y = \sin(\theta)$.

12. A particle is contained inside a box which has orthogonal sides of length a, b and c. The particle's energy E is:
$$E = A\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right),$$
where A is a positive constant. Determine the shape of the box which minimises E, subject to the constraint that the volume of the box V is constant by: (a) using the method of Lagrange multipliers; (b) solving the constraint. Verify that your answers agree.

13. Show that the largest possible volume of a cuboid inscribed in an ellipsoid:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
is $8abc/\sqrt{27}$.

14. The area A of a triangle is given by *Heron's formula* as:
$$A = \sqrt{s(s-a)(s-b)(s-c)},$$
where $s = \frac{1}{2}(a + b + c)$ is the semi-perimeter. Using this formula, show that for a fixed perimeter, the triangle of largest area is equilateral. How does this result change if we additionally demand that the triangle is right-angled, and of fixed perimeter?

15. Find the maximum distance from the origin to the curve $x^2 + y^2 + xy - 4 = 0$.

16. (a) Extremise the function:

$$f(x_1, ..., x_n) = -\sum_{i=1}^{n} x_i \log(x_i), \qquad \text{subject to} \qquad \sum_{i=1}^{n} x_i = 1.$$

(b) Show that if a further constraint:

$$\sum_{i=1}^{n} x_i y_i = Y$$

is applied, where $y_1, ..., y_n$ and Y are given constants, the extremal point of the same function f is located instead where $x_i = a\exp(-by_i)$ where a and b are constants. Write down two equations that determine the values of a and b. [*You need not solve these equations.*]

17. The function:

$$f(n_0, n_1, n_2, ...) = -\sum_{k=0}^{\infty} [n_k \ln(n_k) - n_k]$$

of an infinite number of positive variables is subject to two constraints:

$$\sum_{k=0}^{\infty} n_k = N \qquad \text{and} \qquad \sum_{k=0}^{\infty} E_0 \left(\frac{1}{2} + k\right) n_k = E,$$

where N, E_0 and E are positive constants. Using the method of Lagrange multipliers, show that the stationary point of $f(n_0, n_1, n_2, ...)$ subject to the above constraints occurs when:

$$n_k = 2N \sinh\left(\frac{\beta E_0}{2}\right) e^{-\beta E_0 (1/2+k)}$$

for all k, where β is a Lagrange multiplier. Show further that:

$$E = \frac{NE_0}{2} \coth\left(\frac{\beta E_0}{2}\right).$$