

## Part IA: Mathematics for Natural Sciences B

### Examples Sheet 15: Scalar and vector fields, conservative vector fields, and line integrals

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Questions marked with a (\*) are difficult and should not be attempted at the expense of the other questions.

#### Gradient, directional derivatives, and tangent planes

1. Define the *gradient*  $\nabla f$  of a scalar function. Calculate the gradient of each of the following three-dimensional functions:

$$(a) \, xyz, \quad (b) \, e^{-1/(x+y+z)}, \quad (c) \, e^{-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2},$$

where  $\alpha, \beta, \gamma$  are constants.

2. Let  $f(r)$  be a function only of the spherical radial coordinate  $r = \sqrt{x^2 + y^2 + z^2}$ . Show that  $\nabla f(r) = f'(r)\hat{\mathbf{x}}$ , where  $\mathbf{x} = (x, y, z)$  is the general position vector. Hence evaluate:

$$\nabla(r), \quad \nabla\left(\frac{1}{r}\right), \quad \nabla\left(\frac{e^{-r}}{r}\right).$$

(\*) If you are studying Part IA Physics, what is the physical relevance of the second example? What is the physical relevance of the third example?

3. Let  $\mathbf{a}$  be a fixed vector in  $\mathbb{R}^n$ , and let  $\mathbf{x} = (x_1, \dots, x_n)$  be the general position vector in  $n$  dimensions. Calculate the gradient of each of the following three-dimensional functions:

$$(a) \, \mathbf{a} \cdot \mathbf{x}, \quad (b) \, \mathbf{x} \cdot \mathbf{x}, \quad (c) \, |\mathbf{x}|^{-1}.$$

4. (a) Using the multivariable Taylor expansion, show that for a function  $f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$  of  $n$  variables, we have:

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla f + \dots,$$

where  $\mathbf{x}_0$  is a fixed point, and the gradient  $\nabla f$  is evaluated at  $\mathbf{x}_0$ .

- (b) Hence, give the geometrical meaning of the vector  $\hat{\mathbf{v}} \cdot \nabla f$  for  $\hat{\mathbf{v}}$  a unit vector.  
 (c) Using part (b), explain why  $\nabla f$  points in the direction of greatest increase of a scalar function.
5. (a) Find the rate of change of the function  $f(x, y, z) = \ln(x^2 + y^2) + z$  at the point  $(3, -4, 5)$  in the directions:  
 (i)  $(3, -4, 0)$ ; (ii)  $(1, 2, 0)$ .  
 (b) Find the directional derivative of the function  $f(x, y, z) = \frac{1}{2}[(x-1)^2 + (y-1)^2 + (z-1)^2]$  at the origin  $(0, 0, 0)$  in the direction  $(1, 1, 0)$ .
6. Find the equations of the tangent planes to the following surfaces at the specified points:
- (a)  $z = 3x^2y \sin(\pi x/2)$  at the point  $x = y = 1$ ;  
 (b)  $xz + z^2 - xy^2 = 5$  at the point  $(1, 1, 2)$ .

If you placed a ball on the surface in part (a) at the point  $x = 1, y = 1/2$ , which way would it roll?

### Vector fields and vector derivatives

7. Find the divergence and curl of each of the following vector fields, where  $\mathbf{a}$ ,  $\mathbf{b}$  are constant vectors:

(a)  $\mathbf{x}$ ,    (b)  $\mathbf{a}(\mathbf{x} \cdot \mathbf{b})$ ,    (c)  $\mathbf{a} \times \mathbf{x}$ ,    (d)  $\mathbf{x}/|\mathbf{x}|^3$ ,    (e)  $(\mathbf{a} \cdot \mathbf{b})\mathbf{x} + \mathbf{a}$ ,    (f)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{x})$ .

(\*) Do your expressions for divergence and curl in part (d) make sense at the origin,  $\mathbf{0}$ ?

8. For each of the following pairs of vector fields, calculate  $\nabla \cdot (\mathbf{F} \times \mathbf{G})$  and  $\nabla \times (\mathbf{F} \times \mathbf{G})$ :

(a)  $\mathbf{F} = (x, y, z)$ ,  $\mathbf{G} = (y, -x, 0)$ ;

(b)  $\mathbf{F} = (\sin(x), \sin(y), \sin(z))$ ,  $\mathbf{G} = (\cos(x), \cos(y), \cos(z))$ .

Verify that the identities:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} + \mathbf{F} \cdot (\nabla \times \mathbf{G}),$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

hold in each case, where  $(\mathbf{G} \cdot \nabla)\mathbf{F} = G_x \partial \mathbf{F} / \partial x + G_y \partial \mathbf{F} / \partial y + G_z \partial \mathbf{F} / \partial z$ . [We will learn how to efficiently prove these identities using 'suffix notation' in Part IB Mathematics.]

### Second-order derivatives

9. Define the *Laplacian*  $\nabla^2 \phi$  of a scalar field  $\phi$ . Evaluate the Laplacian of the scalar field  $\phi = \frac{1}{2}(x^2 + y^2 + z^2)$ .

10. Show that for any scalar field  $\phi$  and any vector field  $\mathbf{F}$ , we have  $\nabla \times (\nabla \phi) = \mathbf{0}$  and  $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ .

### Line integrals

11. Let  $f(\mathbf{x})$  be a scalar field and let  $\mathbf{F}(\mathbf{x})$  be a vector field. Suppose that  $\gamma(t)$  is a curve. Give a summary of the steps required to evaluate the following types of line integrals:

$$\int_{\gamma} f(\mathbf{x}) |d\mathbf{x}|, \quad \int_{\gamma} \mathbf{F} \cdot d\mathbf{x}, \quad \int_{\gamma} \mathbf{F} \times d\mathbf{x}.$$

Which of the integrals evaluate to scalars, and which evaluate to vectors? (\*) If you are doing Part IA Physics, suggest a use of each type of line integral.

12. By directly parametrising the curves, evaluate the following line integrals over the specified curves:

(a)  $\int_{C_1} (x^2 + y^2) ds$ ,    (b)  $\int_{C_2} (x^2 + y^2) ds$ ,    (c)  $\int_{C_3} xy ds$ ;

where  $ds = |d\mathbf{x}|$  is the infinitesimal arclength, and:

- $C_1$  is a straight line from  $(0, 0)$  to  $(1, 1)$  in two dimensions;
- $C_2$  is the unit circle centred on the origin in two dimensions;
- $C_3$  is a helix,  $x(t) = \cos(6t)$ ,  $y(t) = \sin(6t)$ ,  $z(t) = 8t$  in three dimension, with  $0 \leq t \leq \pi/12$ .

Give a sketch of the paths of integration in each case.

13. By directly parametrising the curves, evaluate the following line integrals over the specified curves:

$$(a) \int_{C_1} \begin{pmatrix} y \\ -x \\ -1 \end{pmatrix} \cdot d\mathbf{x}, \quad (b) \int_{C_2} \begin{pmatrix} xy \\ x^2 + y \\ 0 \end{pmatrix} \cdot d\mathbf{x}, \quad (c) \int_{C_3} \begin{pmatrix} xy \\ x^2 + y \\ 0 \end{pmatrix} \cdot d\mathbf{x};$$

where:

- $C_1$  is the helix  $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$ , with  $0 \leq t \leq 2\pi$ .
- $C_2$  is the planar curve  $x(t) = t, y(t) = t^2, z(t) = 0$ , with  $0 \leq t \leq 1$ ;
- $C_3$  is the planar curve  $x(t) = 2t - t^2, y(t) = t^4 - 4t^3 + 4t^2, z(t) = 0$ , with  $0 \leq t \leq 1$ ;

Give a sketch of the paths of integration in each case.

14. By directly parametrising the curve, evaluate each of the line integrals:

$$\int_C d\mathbf{x}, \quad \int_C \mathbf{x} \times d\mathbf{x}, \quad \int_C \mathbf{x} \times (\mathbf{x} \times d\mathbf{x}),$$

where  $C$  is the unit circle centred on the origin in the  $x$ - $y$  plane, traversed anticlockwise about the  $z$ -axis. How does the result of the second integral relate to the vector area of the unit disk centred on the origin in the  $x$ - $y$  plane?

### Conservative vector fields

15. State what it means for a vector field  $\mathbf{F}$  to be (a) *conservative*; (b) *irrotational*. How are these conditions related?
16. Show that the following vector fields are conservative by finding potential functions in each case:

$$(a) \mathbf{F}_1 = (yz, xz, xy), \quad (b) \mathbf{F}_2 = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Now, check directly that each of these vector fields is irrotational. (\*) If you are doing Part IA Physics, what is the physical relevance of the second example?

17. Two vector fields are given by  $\mathbf{F}(\mathbf{r}) = \mathbf{a} \times \mathbf{r}$  and  $\mathbf{G} = f(|\mathbf{r}|)\mathbf{r}$ , where  $\mathbf{a}$  is a constant vector,  $\mathbf{r} = (x, y, z)$  is the general position vector, and  $f$  is an arbitrary scalar function. Which, if any, of these functions can be written as the gradient of a scalar function? For those that can, find an appropriate scalar function.
18. Show that  $\mathbf{F}$  is conservative if and only if  $\mathbf{F} \cdot d\mathbf{x}$  is an exact differential, where  $d\mathbf{x} = (dx, dy, dz)$  is an infinitesimal displacement vector.

### Line integrals of conservative vector fields

19. State and prove the *gradient theorem* for line integrals. Using the gradient theorem, show that if  $\mathbf{F}$  is conservative, and  $C_1, C_2$  are paths of integration with the same start and end points, then:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_{C_2} \mathbf{F} \cdot d\mathbf{x}.$$

Deduce also that the line integral of a conservative field around a closed loop is always zero.

20. (a) Without using the gradient theorem, evaluate the integral of the vector field:

$$\mathbf{F}(\mathbf{r}) = e^{-r^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where  $\mathbf{r} = (x, y, z)$  and  $r^2 = x^2 + y^2 + z^2$ , along the curve:

$$\mathbf{r}(t) = \begin{pmatrix} \cos(2\pi t^n) \\ \sin(2\pi t^n) \\ 1 \end{pmatrix},$$

for  $0 \leq t \leq 1$ , and where  $n$  is a fixed positive integer.

- (b) Now, show that  $\mathbf{F}$  is conservative. Hence, use the gradient theorem to verify your answer in part (a) is correct.

21. Consider the following line integrals:

$$\int_{\gamma_i} \mathbf{F} \cdot d\mathbf{x}, \quad \int_{\gamma_i} \mathbf{G} \cdot d\mathbf{x},$$

where:

$$\mathbf{F} = \begin{pmatrix} 4x - 2y - 2z \\ -2x + 2y + az \\ -2x + ay + 2z \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} y \cos(xy) - by + (b+c)z \\ x \cos(xy) + bx - bz \\ (c-b)x + by \end{pmatrix},$$

where  $a, b$  and  $c$  are real constants, and:

- $\gamma_1$  is the set of straight lines connecting the points  $(0, 0, 0)$  to  $(1, 1, 0)$ , followed by  $(1, 1, 0)$  to  $(1, 1, 1)$ ;
- $\gamma_2$  is the curve from  $(0, 0, 0)$  to  $(1, 1, 1)$  on which the position vector is parametrised by  $\mathbf{x}(t) = (t, t, t^2)$  for  $0 \leq t \leq 1$ .

For each value of  $a, b, c$ :

- Evaluate each of the line integrals by directly parametrisating the paths of integration.
- Decide whether  $\mathbf{F}, \mathbf{G}$  are conservative vector fields, and relate this result to the calculation you performed in part (a).