Part IA: Mathematics for Natural Sciences B
Examples Sheet 16: Surface integrals and integral theorems

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Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions.

Surface integrals in cylindrical, spherical, and Cartesian coordinates

1. (**Infinitesimal vector areas**) For each of the following surfaces, draw convincing sketches showing that (for some appropriate choices of orientation of the surfaces):

    (a) On the flat circular surface on the top of a cylinder of radius a, oriented along the z-axis, we have $d\mathbf{S} = \hat{\mathbf{n}} r \, dr \, d\theta$, where $\hat{\mathbf{n}} = (0, 0, 1)$.

    (b) On the curved surface of a cylinder of radius a, oriented along the z-axis, we have $d\mathbf{S} = \hat{\mathbf{n}} a \, d\theta \, dz$, where $\hat{\mathbf{n}} = (a\cos(\theta), a\sin(\theta), 0)$.

    (c) On a sphere of radius a, centred at the origin, we have $d\mathbf{S} = \hat{\mathbf{n}} a^2 \sin(\theta) d\theta d\phi$, where $\hat{\mathbf{n}} = (a\sin(\theta)\cos(\phi), a\sin(\theta)\sin(\phi), a\cos(\theta))$.

    (d) On a plane $\mathbf{r} \cdot \hat{\mathbf{n}} = d$, where $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ is a constant unit normal, we have:

    $$d\mathbf{S} = \frac{\hat{\mathbf{n}}}{n_z} dx dy = \frac{\hat{\mathbf{n}}}{n_y} dx dz = \frac{\hat{\mathbf{n}}}{n_x} dy dz,$$

    depending on whether we choose to parametrise the plane with the coordinates (x, y), (x, z) or (y, z) (noting that when one of the components of $\hat{\mathbf{n}}$ vanishes, such a parametrisation may be impossible).

    How are these infinitesimal vector areas related to the infinitesimal scalar areas, dS?

2. (**Some cylindrical surface integrals**) Evaluate the following surface integrals directly, without using the divergence theorem. Here, $\hat{\mathbf{n}}$ is the unit normal to the surface.

    (a) $\displaystyle\int_S \hat{\mathbf{n}} \cdot d\mathbf{S}$,   (b) $\displaystyle\int_S xy \, dS$,   (c) $\displaystyle\int_S \begin{pmatrix} -yz \\ x^2 \\ 0 \end{pmatrix} \cdot d\mathbf{S}$,   (d) $\displaystyle\int_S \begin{pmatrix} y^2 \\ x^2 \\ 0 \end{pmatrix} \cdot d\mathbf{S}$,

    where:

    (i) S is the unit disc in the $z = 0$ plane, centred on the origin (take the normal in the positive z-direction);

    (ii) S is the curved surface of the cylinder described by $0 \le z \le 1$, $0 \le \theta \le 2\pi$, $r = 1$, in cylindrical polar coordinates (taking the normal pointing in the positive r-direction).

3. (**Some spherical surface integrals**) Evaluate the following surface integrals directly, without using the divergence theorem. In each case, S is the surface of the unit sphere centred at the origin, $\hat{\mathbf{n}}$ is the outward-pointing unit normal, and $\mathbf{r} = (x, y, z)$ is the standard Cartesian position vector.

    (a) $\displaystyle\int_S \hat{\mathbf{n}} \cdot d\mathbf{S}$,   (b) $\displaystyle\int_S xy \, dS$,   (c) $\displaystyle\int_S \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \cdot d\mathbf{S}$,   (d) $\displaystyle\int_S \frac{\sin(\theta)\mathbf{r}}{r} \cdot d\mathbf{S}$,   (e) $\displaystyle\int_S \nabla \times \nabla(x^2 + y^2 + z^2) \cdot d\mathbf{S}$.

4. (**A composite shape**) Without using the divergence theorem, evaluate:

    $$\int_S (z + y^3) dS,$$

    where S is the total surface made from the vertical cylinder $x^2 + y^2 = a^2$ with $0 \le z \le b$, the flat disc $x^2 + y^2 \le a^2$ in the $z = b$ plane ($b > a > 0$), and the hemispherical indentation $x^2 + y^2 + z^2 = a^2$ with $z \ge 0$.

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5. (**Some Cartesian surface integrals**) Evaluate the following surface integrals directly, without using the divergence theorem:

(a) $\displaystyle\int_{S_1} \begin{pmatrix} x^{-1}(\log^2(x)+1)^{-1} \\ e^{-x} \\ z \end{pmatrix} \cdot d\mathbf{S}$,   (b) $\displaystyle\int_{S_2} \begin{pmatrix} |x| \\ 2xy \\ e^{-z} \end{pmatrix} \cdot d\mathbf{S}$,   (c) $\displaystyle\int_{S_3} (x+yz)\, dS$,   (d) $\displaystyle\int_{S_4} \begin{pmatrix} x+y+z \\ (x+y+z)^2 \\ 1-x-y-z \end{pmatrix} \cdot d\mathbf{S}$

where:

- S_1 is the surface of an axes-aligned unit cube, two of whose vertices are at $(0,0,0)$ and $(1,1,1)$;
- S_2 is the surface of an axes-aligned unit cube, two of whose vertices are at $(1,0,0)$ and $(2,1,1)$;
- S_3 is the surface $0 \leq x \leq 2\pi, 0 \leq z \leq 1$ and $y = 1+z$;
- S_4 is the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

6. (**A more abstract surface integral**) Let $\mathbf{a}$, $\mathbf{b}$ be constants, and let $\mathbf{r} = (x,y,z)$ be the standard Cartesian position vector. Without using the divergence theorem, calculate the flux of the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{r}$ through:

(a) the triangle OAB where O denotes the origin (where A has position vector $\mathbf{a}$ and B has position vector $\mathbf{b}$);

(b) the closed hemisphere with curved surface and base given by $x^2+y^2+z^2 = R^2, 0 \leq z \leq R$ and $x^2+y^2 \leq R^2$, $z = 0$, respectively, where the parameter $R > 0$.

Surface integrals in general coordinate systems

7. Suppose that a surface S is described as the set of all position vectors $\mathbf{r}(u,v)$ as the parameters u, v vary. Show that the infinitesimal vector area on the surface near the point with parameters (u,v) is given by:

$$d\mathbf{S} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\,dv.$$

Hence, rederive the formulae obtained in Question 1 algebraically.

8. Consider the semi-infinite cone described by the equation $bz = ab - \sqrt{x^2+y^2}$.

(a) Show that the infinitesimal vector area element on the surface of the cone may be expressed equivalently in terms of Cartesian coordinates (x,y) or cylindrical polar coordinates (r,θ) via:

$$d\mathbf{S} = \begin{pmatrix} \frac{ax}{b\sqrt{x^2+y^2}} \\ \frac{ay}{b\sqrt{x^2+y^2}} \\ 1 \end{pmatrix} dx\,dy = \begin{pmatrix} a\cos(\theta)/b \\ a\sin(\theta)/b \\ 1 \end{pmatrix} r\,dr\,d\theta.$$

(b) Hence, evaluate the surface area of a cone of height a, with base of radius b, using (i) Cartesian coordinates; (ii) cylindrical polar coordinates.

(c) Find also the flux of the vector field $\mathbf{F} = (y,x,1)$ through the curved surface of the cone with $z \geq 0$.

9. Consider the paraboloid described by the equation $b^2 z = ab^2 - x^2 - y^2$.

(a) Show that the infinitesimal vector area element on the surface of the paraboloid may be expressed equivalently in terms of Cartesian coordinates (x,y) or cylindrical polar coordinates (r,θ) via:

$$d\mathbf{S} = \begin{pmatrix} 2ax/b^2 \\ 2ay/b^2 \\ 1 \end{pmatrix} dx\,dy = \begin{pmatrix} 2ar\cos(\theta)/b^2 \\ 2ar\sin(\theta)/b^2 \\ 1 \end{pmatrix} r\,dr\,d\theta.$$

(b) Hence, evaluate the surface area of a paraboloid of height a, with base of radius b, using (i) Cartesian coordinates; (ii) cylindrical polar coordinates.

(c) Find also the flux of the vector field $\mathbf{F} = (y,x,1)$ through the curved surface of the paraboloid with $z \geq 0$.

The divergence theorem

10. Carefully state the *divergence theorem*. You should define all symbols that arise and specify orientations carefully.

11. Check explicitly that the divergence theorem holds for:

    (a) the flux of the vector field $\mathbf{F} = (x^2 + y^2, 3xy, 6z)$ through the surface of the unit cube $0 \le x \le 1, 0 \le y \le 1$ and $0 \le z \le 1$;

    (b) the flux of the vector field $\mathbf{F} = (x^3, y^3, z^3)$ through the surface of the unit sphere centred on the origin;

    (c) the flux of the vector field $\mathbf{F} = (0, (y+2x-4)^2, 1-z^2)$ through the surface of the triangular prism, $0 \le z \le 1$, whose base in the xy-plane has vertices $(0,0,0), (2,0,0), (0,4,0)$.

12. Evaluate: (a) $\int_S \mathbf{r} \cdot d\mathbf{S}$, (b) $\int_V \nabla \cdot \hat{\mathbf{r}} \, dV$, where V is the unit sphere, S is the surface of the unit sphere, and $\mathbf{r} = (x, y, z)$ is the standard Cartesian position vector.

13. Evaluate:
$$\int_S \begin{pmatrix} 0 \\ (y+3)^2 \\ z^2 \end{pmatrix} \cdot d\mathbf{S},$$
where S is the surface of the pyramid bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z/2 = 1$.

14. (a) Starting from the Maxwell equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, where $\mathbf{E}(\mathbf{x}, t)$ is the electric field, $\rho(\mathbf{x}, t)$ is charge density, and ϵ_0 is a constant called the *permittivity of free space*, show that $\Phi_E = Q/\epsilon_0$, where Φ_E is the electric flux through a closed surface S, and Q is the total charge contained in the volume bounded by that surface. This result is called *Gauss' law for electric fields*.

    (b) Another of the Maxwell equations states that $\nabla \cdot \mathbf{B} = 0$, where $\mathbf{B}(\mathbf{x}, t)$ is the magnetic field. What does the divergence theorem tell you about the magnetic flux through a closed surface S?

Stokes' theorem

15. Carefully state *Stokes' theorem*. You should define all symbols that arise and specify orientations carefully.

16. Check explicitly that Stokes' theorem holds for:

    (a) the surface integral of the curl of $\mathbf{F} = (yz, zx, xy)$ through the disk $x^2 + y^2 \le 1, z = 0$;

    (b) the surface integral of the curl of $\mathbf{F} = (y, z, x)$ through the hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$;

    (c) the surface integral of the curl of $\mathbf{F} = (yz, zx, xy)$ through the paraboloid $z = 4 - x^2 - y^2, z \ge 0$.

17. Using Stokes' theorem, evaluate the line integral of $\mathbf{F} = (z, x, y)$ over the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, oriented anticlockwise as seen from above.

18. Starting from the Maxwell equation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, where $\mathbf{E}(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t)$ are the electric and magnetic field respectively, show that $\mathcal{E} = -d\Phi_B/dt$, where Φ_B is the magnetic flux through a time-independent surface S, and $\mathcal{E}$ is the *electromotive force* around the boundary C, which you should find an expression for in terms of a line integral. This result is called *Faraday's law of induction*.

19. ($*$) Using Stokes' theorem, prove that:
$$\frac{1}{2} \oint_C d\mathbf{x} \times \mathbf{x} = \int_S d\mathbf{S},$$
where C is the boundary of the surface S. Deduce that the vector area of a closed surface is necessarily zero.