Part IA: Mathematics for Natural Sciences B
Examples Sheet 3: Complex numbers and hyperbolic functions

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Real and imaginary parts

1. Find the real and imaginary parts of the following numbers (where n is an integer):

   (a) i^3, (b) i^{4n}, (c) $\left(\frac{1+i}{\sqrt{2}}\right)^2$, (d) $\left(\frac{1-i}{\sqrt{2}}\right)^2$, (e) $\left(\frac{1+\sqrt{3}i}{2}\right)^3$, (f) $\frac{1+i}{2-5i}$, (g) $\left(\frac{1+i}{1-i}\right)^2$.

2. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:

   (a) z^2, (b) iz, (c) $(1+i)z$, (d) $z^2(z-1)$, (e) $z^*(z^2 - zz^*)$.

3. Define u and v to be the real and imaginary parts, respectively, of the complex function $w = 1/z$. Show that the contours of constant u and v are circles. Show also that the contours of u and the contours of v intersect at right angles.

Factoring polynomials and solving equations

4. Factorise the following expressions: (a) $z^2 + 1$; (b) $z^2 - 2z + 2$; (c) $z^2 + i$; (d) $z^2 + (1-i)z - i$. *[Hint: you have already computed the two square roots of i in Question 1(c).]*

5. Given that $z = 2 + i$ solves the equation $z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0$, find the remaining solutions.

6. Consider the polynomial equation $a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0$, where the coefficients $a_n, a_{n-1}, ..., a_0$ are real. Show that the solutions to this equation come in complex conjugate pairs. Deduce that if n is odd, there is at least one real solution.

Geometry of complex numbers

7. Using a diagram, explain the geometric meaning of the *modulus*, $|z|$, and *argument*, $\arg(z)$, of a complex number z. Find the moduli and (principal) arguments of: (a) $1 + \sqrt{3}i$; (b) $-1 + i$; (c) $-\sqrt{3} - i/\sqrt{3}$.

8. For $z \in \mathbb{C}$, show that $|z|^2 = zz^*$. Hence prove that $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$, where $a, b \in \mathbb{C}$, and interpret this result geometrically. *[Hint: you don't need to split a, b into real and imaginary parts.]*

9. By writing $z = |z|(\cos(\arg(z)) + i\sin(\arg(z)), w = |w|(\cos(\arg(w)) + i\sin(\arg(w))$, compute the modulus and argument of the product zw. Hence give the geometrical interpretation of multiplying one complex number by another complex number. Give also a geometrical interpretation of division of one complex number by another complex number, z/w.

10. Let $z_1 = 2 + i$, $z_2 = 3 + 4i$. Find $z_1 z_2$ by: (a) adding arguments and multiplying moduli; (b) using the rules of complex algebra. Verify that your results agree.

11. By considering multiplication of the complex numbers $z = 1 + iA$ and $w = 1 + iB$, derive the arctangent addition formula:

$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right).$$

12. Give a geometrical interpretation (in terms of *vectors*) of the real and imaginary parts of the quantity $Q = z_1 z_2^*$. Show also that Q is invariant under a rotation of z_1, z_2 about the origin, and confirm that this is consistent with your geometrical interpretation. *[Hint: In Question 9, you showed that multiplying by a complex number u of unit modulus is equivalent to a rotation about the origin.]*

Loci in the complex plane

13. (**Circles**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

    (a) $|z| = 4$, (b) $|z-1| = 3$, (c) $|z-i| = 2$, (d) $|z-(1-2i)| = 3$, (e) $|z^*-1| = 1$, (f) $|z^*-i| = 1$.

14. (**Transformations of circles**) Describe the set of points $z \in \mathbb{C}$ satisfying $|z-2-i| = 6$. Without further calculation, describe the sets of points $u \in \mathbb{C}, v \in \mathbb{C}, w \in \mathbb{C}$ satisfying:

    (a) $u = z + 5 - 8i$, (b) $v = iz + 2$, (c) $w = \frac{3}{2}z + \frac{1}{2}z^*$,

    where $|z-2-i| = 6$.

15. (**Circles of Apollonius**) Let $a, b \in \mathbb{C}$. Show that the set of points satisfying $|z-a| = \lambda|z-b|$, where $\lambda \neq 1$, is a circle in the complex plane. *[Hint: start by squaring the equation. You don't need to split z into real and imaginary parts.]* Determine the centre and radius of the circle $|z| = 2|z-2|$.

16. (**Lines and half-lines**) Describe the sets of points $z \in \mathbb{C}$ satisfying:

    (a) $|z-2| = |z+i|$, (b) $|z-2| = |z^*+i|$, (c) $\arg(z) = \pi/2$, (d) $\arg(z^*) = \pi/4$.

17. (**Lines and circles**) Let $a \in \mathbb{R}$ and $b, c \in \mathbb{C}$. *Without setting* $z = x + iy$, describe the locus $azz^* + bz + b^*z^* + c = 0$ for different values of a, b, c. How does the locus change under the maps: (a) $z \mapsto \alpha z$ for $\alpha \in \mathbb{C}$; (b) $z \mapsto 1/z$?

18. (**More complex figures**) Sketch the sets of points $z \in \mathbb{C}$ satisfying:

    (a) $\mathrm{Re}(z^2) = \mathrm{Im}(z^2)$, (b) $\dfrac{\mathrm{Im}(z^2)}{z^2} = -i$, (c) $|z^* + 2i| + |z| = 4$, (d) $|2z - z^* - 3i| = 2$.

Exponential form of a complex number

19. State *Euler's formula* for the complex exponential $e^{i\theta}$. Hence provide a simpler derivation of the modulus-argument multiplication law proved in Question 9.

20. Find (a) the real and imaginary parts; (b) the modulus and argument, of:

    $$\frac{e^{i\omega t}}{R + i\omega L + (i\omega C)^{-1}},$$

    where ω, t, R, L, C are real, quoting your answers in terms of $X = \omega L - (\omega C)^{-1}$. ($*$) If you are taking IA Physics, can you think of what each of ω, t, R, L, C might represent?

21. Express each of the following in Cartesian form: (a) $e^{-i\pi/2}$; (b) $e^{-i\pi}$; (c) $e^{i\pi/4}$; (d) e^{1+i}; (e) $e^{2e^{i\pi/4}}$.

22. Let a, b, ω be real constants. Show that $a\cos(\omega x) + b\sin(\omega x) = \mathrm{Re}((a-bi)e^{i\omega x})$, and hence, by writing $a - bi$ in exponential form, deduce that $a\cos(\omega x) + b\sin(\omega x) = \sqrt{a^2+b^2}\cos(\omega x - \arctan(b/a))$.

Multi-valued functions: logarithms and powers

23. Explain why the complex logarithm $\log : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is a *multi-valued function*, and give its possible values. Using the complex logarithm, find all complex numbers satisfying: (a) $e^{2z} = -1$; (b) $e^{z^*} = i + 1$.

24. Let the real and imaginary parts of the complex logarithm $\log(z)$ be u, v respectively. Sketch the contours of constant u, v in the complex plane, and show that they intersect at right angles.

25. Find the real and imaginary parts of the function $f(z) = \log(z^{1+i})$. Hence, sketch the locus $\mathrm{Re}(f(z)) = 0$.

26. Explain how the complex logarithm can be used to define complex powers, z^w, and hence describe the multi-valued nature of complex exponentiation. Compute all values of the multi-valued exponentials: (a) i^i; (b) $i^{1/3}$.

27. Compute all possible values of $(i^i)^i$ and $i^{(i^i)}$.

Roots of unity

28. Write down the solutions to the equation $z^n = 1$ in terms of complex exponentials, and plot the solutions on an Argand diagram. *[Recall that the solutions are called the* nth *roots of unity.]*

29. Find and plot the solutions to the following equations: (a) $z^3 = -1$; (b) $z^4 = 1$; (c) $z^2 = i$; (d) $z^3 = -i$.

30. If $\omega^n = 1$, determine the possible values of $1 + \omega + \omega^2 + \cdots + \omega^{n-1}$, and interpret your result geometrically.

31. Show that the roots of the equation $z^{2n} - 2bz^n + c = 0$ will, for general complex values of b and c and integral values of n, lie on two circles in the Argand diagram. Give a condition on b and c such that the circles coincide. Find the largest possible value for $|z_1 - z_2|$, if z_1 and z_2 are roots of $z^6 - 2z^3 + 2 = 0$.

Trigonometry with complex numbers

32. Prove *De Moivre's formula*, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$. Hence, solve the equation $16\sin^5(\theta) = \sin(5\theta)$ by expressing $\sin(5\theta)$ in terms of $\sin(\theta)$ and its powers.

33. Starting from Euler's formula, show that the trigonometric functions can be written in terms of complex exponentials as:
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$
Learn these formulae off by heart. Hence, express $\sin^5(\theta)$ in terms of $\sin(\theta), \sin(3\theta)$ and $\sin(5\theta)$.

34. Show that if $x, y \in \mathbb{R}$, the equation $\cos(y) = x$ has the solutions $y = \pm i\log\left(x + i\sqrt{1-x^2}\right) + 2n\pi$ for integer n.

35. Find the real and imaginary parts of the function $\tan(z^*)$.

36. Let $\theta \neq 2p\pi$ for $p \in \mathbb{Z}$. Show that $\displaystyle\sum_{n=0}^{N-1} \cos(n\theta) = \frac{\cos\left((N-1)\theta/2\right)\sin\left(N\theta/2\right)}{\sin\left(\theta/2\right)}$. What happens if $\theta = 2p\pi$?

Hyperbolic functions

37. (a) Give the definitions of $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.
    (b) Hence, show that $\cos(x) = \cosh(ix)$ and $i\sin(x) = \sinh(ix)$. Deduce *Osborn's rule*: 'a hyperbolic trigonometric identity can be deduced from a circular trigonometric identity[1] by replacing each trigonometric function with its hyperbolic counterpart *except* where sine enters quadratically, where we include an extra factor of -1.'
    (c) Using Osborn's rule, write down the formula for $\tanh(x + y)$ in terms of $\tanh(x), \tanh(y)$.

38. Find the real and imaginary parts of the following complex numbers:

    (a) $\log\left[\sinh\left(\frac{i\pi}{2}\right) + \cosh\left(\frac{9i\pi}{2}\right)\right]$, (b) $\displaystyle\sum_{n=1}^{121}\left[\tanh\left(\frac{in\pi}{4}\right) - \tanh\left(\frac{in\pi}{4} - \frac{i\pi}{4}\right)\right]$.

39. Let $b \geq a > 0$ be fixed, and let θ be a variable parameter. Find the Cartesian equations of the two parametric curves: (a) $(x, y) = (a\cos(\theta), b\sin(\theta))$; (b) $(x, y) = (a\cosh(\theta), b\sinh(\theta))$, and sketch them in the plane. [*This explains why hyperbolic functions are called hyperbolic functions!*]

40. Sketch the graphs of $\cosh(x), \sinh(x)$ and $\tanh(x)$, noting any asymptotes. Hence, sketch the graphs of $\cosh^{-1}(x)$, $\sinh^{-1}(x)$ and $\tanh^{-1}(x)$.

41. Express $\cosh^{-1}(x), \sinh^{-1}(x)$ and $\tanh^{-1}(x)$ as logarithms, justifying any sign choices you make.

42. Solve the equation $\cosh(x) = \sinh(x) + 2\text{sech}(x)$, giving the solutions as logarithms.

43. Find all solutions to the equations: (a) $\cosh(z) = i$; (b) $\sinh(z) = -2$; (c) $\tanh(z) = -i$.

[1] Provided the arguments of all the circular trigonometric functions are homogeneous linear polynomials in the variables of interest.