

## Part IA: Mathematics for Natural Sciences B

### Examples Sheet 4: Differential calculus, limits and continuity

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Questions marked with a (\*) are difficult and should not be attempted at the expense of the other questions. A section marked with a (†) contains content that is unique to the Mathematics B course.

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#### Limit definition of the derivative

- Let  $y \equiv y(x)$  be a function of  $x$ . Define the *derivative*  $dy/dx$  of  $y$  as a limit. Using the limit definition:
  - show that differentiation is a linear operation;
  - find the derivative of  $y(x) = x^n$ , for  $n = 0, 1, 2, 3, \dots$Hence obtain the derivative of  $ax + bx^2 \sin(\theta)$ , where  $a, b, \theta$  are real constants.
- Using *only* the limit definition, show that for  $a > 0$ , the derivative of  $y(x) = a^x$  is proportional to  $a^x$ .
  - One definition of the number  $e$  is the value of  $a$  for which the proportionality constant in the previous part is 1. Using *only* this definition, show that the derivative of  $a^x$  is given by  $\log(a)a^x$ .

#### Rules of differentiation

- Let  $y \equiv y(x)$ ,  $u \equiv u(x)$  and  $v \equiv v(x)$  be functions of  $x$ . Using the limit definition of the derivative, prove the following rules of differentiation:

$$(a) \frac{d}{dx}(u(v)) = \frac{dv}{dx} \frac{du}{dv}, \quad (b) \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}, \quad (c) \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}, \quad (d) \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}.$$

[Recall that these rules are called the *chain rule*, the *product rule*, the *quotient rule*, and the *reciprocal rule*, respectively. Make sure you know them off by heart!] Rewrite these rules in terms of Lagrange's 'primed' notation for derivatives.

- Using the rules you derived in the previous question, compute the derivatives of:

$$(a) \log(x), \quad (b) 3^{x^2}, \quad (c) \frac{e^x}{x^3 - 1}, \quad (d) x^3 \log(x^2 - 7), \quad (e) \sqrt{x^3 - e^x \log(x)}.$$

- By writing each of the following trigonometric and hyperbolic functions in terms of exponentials, compute their derivatives: (a)  $\cos(x)$ ; (b)  $\sin(x)$ ; (c)  $\cosh(x)$ ; (d)  $\sinh(x)$ ; (e)  $\tan(x)$ ; (f)  $\tanh(x)$ . Learn these derivatives off by heart.
- Using: (i) the logarithmic formulae for the inverse hyperbolic functions you derived on Sheet 3; (ii) the reciprocal rule, compute the derivatives of: (a)  $\cosh^{-1}(x)$ ; (b)  $\sinh^{-1}(x)$ ; (c)  $\tanh^{-1}(x)$ . Learnt these derivatives off by heart.
- If  $y \equiv y(x)$  is a function of  $x$ , show that  $\frac{d^3x}{dy^3} = -\left(\frac{dy}{dx}\right)^{-4} \frac{d^3y}{dx^3} + 3\left(\frac{dy}{dx}\right)^{-5} \left(\frac{d^2y}{dx^2}\right)^2$ . Verify this when  $y = e^{2x}$ .
- What is *implicit differentiation*, and why is it called implicit? Using: (a) implicit differentiation; (b) the reciprocal rule, find  $dy/dx$  given  $y + e^y \sin(y) = 1/x$ , and make sure that your answers agree.

#### Curve-sketching

- State what it means for a function to be *even* and for a function to be *odd*, and explain the geometric significance of these definitions. Hence, decide whether the following functions are even, odd, both, or neither:

$$(a) x, \quad (b) \sin(x), \quad (c) e^x, \quad (d) \sin\left(\frac{\pi}{2} - x\right), \quad (e) |x| \cos(x), \quad (f) \sqrt{x}, \quad (g) 2, \quad (h) 0, \quad (i) \log\left|\frac{1+x}{1-x}\right|.$$

10. Write down a list of things you should consider when sketching the graph of a function. Compare with your supervision partner before the supervision, and exchange ideas!
11. Sketch the graphs of the following functions, explaining your reasoning in each case:

$$(a) (x-3)^3 + 2x, \quad (b) \frac{x}{1+x^2}, \quad (c) \frac{x^2+3}{x-1}, \quad (d) xe^x, \quad (e) \frac{\log(x)}{1+x}, \quad (f) \frac{1}{1-e^x}, \quad (g) e^x \cos(x).$$

(†) **Leibniz's formula**

12. Using mathematical induction, prove *Leibniz's formula* for the  $n$ th derivative of a product:

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)},$$

where  $f^{(k)}$  denotes the  $k$ th derivative of  $f$ . Hence compute: (a) the third derivative of  $\log^2(x)$ ; (b) the 100th derivative of  $x^2 e^x$ .

13. Use Leibniz's formula to prove that the  $n$ th derivative of  $e^{-x^2/2}$  is a solution of the equation  $Z'' + xZ' + (n+1)Z = 0$ .

(†) **Formal definition of a limit**

14. Suppose that  $f : (a, b) \setminus \{x_0\} \rightarrow \mathbb{R}$  is a real function defined on a (possibly infinite) open interval excluding a point  $x_0 \in \mathbb{R}$ . Give the formal mathematical definition of the phrase ' $f(x) \rightarrow l$  as  $x \rightarrow x_0$ ', and explain this definition using a diagram. How should this definition be modified for the cases  $l = \pm\infty$ ?
15. Here is a model example of a formal mathematical argument, from first principles, showing that  $x^2 \rightarrow 1$  as  $x \rightarrow 1$ :

*'Suppose we are given some arbitrary tolerance  $\epsilon > 0$ . Choose some closeness  $\delta = \min(1, \epsilon/3)$ . Then for all  $x$  which are  $\delta$ -close to 1, i.e.  $0 < |x - 1| < \delta$ , we have:*

$$|x^2 - 1| = |((x - 1) + 1)^2 - 1| \tag{1}$$

$$= |(x - 1)^2 + 2(x - 1)| \tag{2}$$

$$\leq |x - 1|^2 + 2|x - 1| \tag{3}$$

$$< \delta^2 + 2\delta \tag{4}$$

$$\leq \delta + 2\delta \tag{5}$$

$$= 3\delta \tag{6}$$

$$\leq \epsilon. \tag{7}$$

*Hence if  $x$  is  $\delta$ -close to 1, we have that  $|x^2 - 1| < \epsilon$ , so that  $x^2$  is  $\epsilon$ -close to 1. We conclude that, by the definition of a limit, we have  $x^2 \rightarrow 1$  as  $x \rightarrow 1$ .'*

- (a) Which of  $\epsilon, \delta$  are we given, and which of  $\epsilon, \delta$  must we choose?
- (b) Why do we express  $|x^2 - 1|$  in terms of  $x - 1$  in line (1)?
- (c) What law from earlier in the course have we used in going from line (2) to line (3)?
- (d) What have we used in going from line (4) to line (5)? What about in going from line (6) to line (7)?
- (e) Would the proof still have been successful if we had chosen  $\delta = \min(1, \epsilon/4)$ ? In terms of  $\epsilon$ , what is the largest possible value of  $\delta$  we could have chosen for the proof to still work?

16. Using the model example in Question 14 as a template, provide proofs from first principles showing that:

- (a)  $4x^3 \rightarrow 0$  as  $x \rightarrow 0$ , (b)  $x^2 \rightarrow a^2$  as  $x \rightarrow a$ , for  $a \in \mathbb{R}$ , (c)  $\sin(x) \rightarrow 1$  as  $x \rightarrow \pi/2$ ,  
 (d)  $x \sin(1/x) \rightarrow 0$  as  $x \rightarrow 0$ , (e)  $1/x^2 \rightarrow \infty$  as  $x \rightarrow 0$ .

17. Suppose that  $f : (a, \infty) \rightarrow \mathbb{R}$  is a real function defined on an open interval up to positive infinity. Give the formal mathematical definition of the phrase ' $f(x) \rightarrow l$  as  $x \rightarrow \infty$ ', and explain this definition using a diagram. How should this definition be modified for the cases  $l = \pm\infty$ ? Hence, show directly from the definition that: (a)  $1/x \rightarrow 0$  as  $x \rightarrow \infty$ ; (b)  $\sin(x)/x \rightarrow 0$  as  $x \rightarrow \infty$ ; (c)  $x^3 \rightarrow -\infty$  as  $x \rightarrow -\infty$ .

### (†) Laws of limits

18. (\*) Let  $x_0 \in \mathbb{R}$ , and let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be real functions. From the formal definition of a limit, prove that:

$$\lim_{x \rightarrow x_0} (f(x) + g(x)) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x),$$

provided that (i) both the limits on the right hand side exist in  $\mathbb{R} \cup \{\infty, -\infty\}$  (the set of real numbers with infinity and negative infinity adjoined), and (ii) if one of the limits on the right hand side is  $\infty$ , the other is *not*  $-\infty$ .

19. From the formal mathematical definition of a limit, it is possible to prove results about the limits of sums, products, quotients and compositions of functions, similarly to Question 18. State these '*laws of limits*' clearly (making sure to take particular care when the limits are infinite), and use them to evaluate the following:

(a)  $\lim_{x \rightarrow 0} \frac{x+1}{2-x^2}$ , (b)  $\lim_{x \rightarrow \infty} \sin\left(\frac{x^2+x+1}{3x^2-4}\right)$ , (c)  $\lim_{x \rightarrow 0} \left(\exp\left(\frac{x^4-1}{x^4+1}\right)\right)^{1/x^2}$ , (d)  $\lim_{x \rightarrow \infty} (\sqrt{x^2+7x}-x)$ .

20. State L'Hôpital's rule for evaluating limits of differentiable functions, carefully specifying the conditions under which it is valid. Assuming  $\alpha > 0$  throughout, use L'Hôpital's rule - where appropriate - to evaluate the limits of the following functions both (i) as  $x \rightarrow 0^+$  (a *one-sided limit*), and (ii) as  $x \rightarrow \infty$ :

(a)  $x^\alpha \log(x)$ , (b)  $x^{-\alpha} \log(x)$ , (c)  $x^\alpha e^{-x}$ , (d)  $x^{-\alpha} e^x$ , (e)  $\sin(\alpha x)/x$ .

21. Using L'Hôpital's rule, evaluate the following 'power law' limits:

(a)  $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x$ , (b)  $\lim_{x \rightarrow \infty} \log^{1/x}(x)$ , (c)  $\lim_{x \rightarrow 0^+} x^x$ , (d)  $\lim_{x \rightarrow \infty} x^{1/x}$ .

22. Explain why the following arguments with limits are *wrong*.

(a)  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} (1+0)^x = 1$ , using  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$  in the first step.

(b)  $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \rightarrow 0} \frac{1-1}{x^2} = 0$ , using  $\cos(x) \approx 1$  for small enough  $x$ .

Now, evaluate the limits correctly.

23. Using L'Hôpital's rule where appropriate, compute the limit:

$$\lim_{x \rightarrow \infty} \left(1 + a^x + \left(\frac{a^2}{2}\right)^x\right)^{\frac{1}{x}}.$$

for all values of  $a \geq 0$ .

24. Consider the limit:

$$\lim_{x \rightarrow \infty} \frac{x}{x + \sin(x)}.$$

Show that this limit is equal to one. Show that if we instead naïvely apply L'Hôpital's rule, we incorrectly conclude that the limit does not exist.

**(†) Miscellaneous limits**

[This section contains a large collection of limits from past papers for you to evaluate. If you feel like you are getting too much of a good thing, feel free to save some of them for us to do together in the supervision.]

25. Evaluate the following limits, using the most efficient method in each case:

- |   |   |
|---|---|
| (a) $\lim_{x \rightarrow 0^+} x \log(x);$                           | (b) $\lim_{x \rightarrow a} \frac{x^x - a^a}{x - a}$ where $a > 0;$ |
| (c) $\lim_{x \rightarrow 0} \frac{\tan(x) - \sin(x)}{\sin^3(x)};$   | (d) $\lim_{x \rightarrow a} \frac{\sin(x) - \sin(a)}{x - a};$       |
| (e) $\lim_{x \rightarrow \infty} \left( \frac{x+a}{x-a} \right)^x;$ | (f) $\lim_{x \rightarrow 0} \frac{\cos(x) - \cos(3x)}{x^2};$        |
| (g) $\lim_{x \rightarrow 0} \frac{\log(\cos(x))}{\log(\cos(3x))};$  | (h) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{\sinh(x)}.$             |

**(†) Continuity of functions**

26. Let  $f : (a, b) \rightarrow \mathbb{R}$  be a real function, and let  $x_0 \in (a, b)$  be a point in its domain.

- (a) State the formal  $\epsilon, \delta$  definition of  $f$  being *continuous* at  $x_0$ . Explain this condition by drawing a diagram.  
(b) Using the formal definition of a limit, explain why this condition is equivalent to the statement:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

27. Using the formal  $\epsilon, \delta$  definition of continuity, show directly that the following functions are continuous everywhere:

- (a)  $x$ ,                      (b)  $|x|$ ,                      (c)  $x^2$ ,                      (d)  $\sin(x)$ .

At what points are these functions differentiable? [Hint: for part (c), look at your answer to Question 16(b).]

28. Using the formal  $\epsilon, \delta$  definition of continuity, show directly that the function  $f(x) = 0$  for  $x \leq 0$ ,  $f(x) = 1$  for  $x > 0$ , is discontinuous at  $x = 0$ .  
29. Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x \sin(1/x)$  for  $x \neq 0$ , and  $f(0) = 0$ . Show that  $f$  is continuous everywhere, and is differentiable everywhere except at  $x = 0$ .  
30. Consider the functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  defined by:

$$f(x) = \begin{cases} |x|^p \sin(x), & x \neq 0, \\ 0, & x = 0, \end{cases} \quad g(x) = \begin{cases} |x|^q \sin(\pi \sin(1/x)), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where  $p, q$  are real numbers. For which values of  $p, q$  are  $f, g$ : (a) continuous; (b) differentiable? Justify your answers.

31. Three functions  $f_0, f_1, f_2$  are defined by:

$$f_n(x) = \left( \frac{x - \pi/2}{x} \right)^n \sin(\tan(x))$$

for  $n = 0, 1, 2$ , at all points except  $x = m\pi/2$  for integer  $m$ , where the functions are defined to be zero. For each  $n$ , determine with justification all points in the range  $(-\pi, \pi)$  where the function is: (a) continuous; (b) differentiable.

32. Show that if a function is differentiable at a point  $x_0$  in its domain, then it must be continuous at  $x_0$ . (\*) Is it true that a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  must be differentiable at *some* point?