# Part IA: Mathematics for Natural Sciences B
# Examples Sheet 6: Single-variable integration

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Questions marked with a (∗) are difficult and should not be attempted at the expense of the other questions. A section marked with a (†) contains content that is unique to the Mathematics B course.

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### Riemann sums and the definition of the integral

1. Explain what is meant by a *Riemann sum* for a function $f : [a,b] \to \mathbb{R}$ using a *partition* $P = (x_0, ..., x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, ..., t_n)$. By choosing appropriate partitions and taggings in each case, use sequences of Riemann sums to evaluate the definite integrals of the following functions on $[0,1]$ from first principles:

   (a) $x$, (b) $x^2$, (c) $x^3$, (d) $\sqrt{x}$, (e) $\cos(x)$.

   [*Hint: For part (d), consider a non-uniform tagging. For part (e), consider the integral of* $\mathrm{Re}(e^{ix})$ *instead of* $\cos(x)$.]

2. Using a non-uniform tagging, use a sequence of Riemann sums to evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{x^{1+\alpha}}$, where $\alpha > 0$.

3. Assuming standard integrals, show by considering Riemann sums that $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2} = \frac{\pi}{4}$.

4. (∗) If a sequence of Riemann sums for a function $f : [a,b] \to \mathbb{R}$ converges, must the function be integrable?

### Basic integrals

5. *Write down* the indefinite integrals of each of the following functions, where $a \neq 0$, $\alpha \neq -1$, and $f$ is any (differentiable, non-zero) function:

   (a) $(ax+b)^\alpha$, (b) $e^{ax+b}$, (c) $(ax+b)^{-1}$, (d) $\sin(ax+b)$, (e) $\cos(ax+b)$,

   (f) $\sec^2(ax+b)$, (g) $\mathrm{cosec}^2(ax+b)$, (h) $\sinh(ax+b)$, (i) $\cosh(ax+b)$, (j) $f'(x)f(x)^\alpha$,

   (k) $f'(x)/f(x)$.

   Learn these integrals off by heart, and get your supervision partner to test you on them.

6. Using the results of the previous question, evaluate the definite integrals:

   (a) $\displaystyle\int_0^2 (x-1)^2\,dx$, (b) $\displaystyle\int_0^\pi e^{i\theta}\,d\theta$, (c) $\displaystyle\int_0^\pi \cos(x)\,dx$, (d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$, (e) $\displaystyle\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx$.

7. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax}\cos(bx)$. Similarly, determine the indefinite integrals of $e^x(\sin(x)-\cos(x))$ and $e^x(\sin(x)+\cos(x))$.

### Integration by substitution

8. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

   (a) $\dfrac{1}{\sqrt{1-x^2}}$, (b) $\dfrac{1}{\sqrt{x^2-1}}$, (c) $\dfrac{1}{\sqrt{1+x^2}}$, (d) $\dfrac{1}{1+x^2}$, (e) $\dfrac{1}{1-x^2}$

   Learn these integrals off by heart, and get your supervision partner to test you on them.

9. Using the results of the previous question, determine: (a) $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}}$; (b) $\displaystyle\int \frac{8-2x}{\sqrt{6x-x^2}}\,dx$.

10. By means of an appropriate substitution in each case, determine the indefinite integrals of the following functions:

(a) $x\sqrt{x+3}$, (b) $\tan(x)\sqrt{\sec(x)}$, (c) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$, (d) $\dfrac{1}{x\sqrt{x^2-1}}$.

11. This question shows that any trigonometric integral can be turned into an algebraic integral through the use of the powerful *half-tangent substitution*.

    (a) Show that if $t = \tan\left(\frac{1}{2}x\right)$, then $\sin(x) = 2t/(1+t^2)$, $\cos(x) = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$. Deduce that for any function $f$, we have:

$$\int f\left(\sin(x), \cos(x)\right)\,dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

    (b) Using the method derived in (a), find the indefinite integrals of the following functions:

(i) $\operatorname{cosec}(x)$, (ii) $\sec(x)$, (iii) $\dfrac{1}{2+\cos(x)}$.

**Partial fractions and rational functions**

12. Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

(a) $\dfrac{1}{1-x^2}$, (b) $\dfrac{3x}{2x^2+x-1}$, (c) $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}$.

Compare your answer to (a) with your answer to Question 7(e), where you evaluated the same integral using a substitution. Are your results compatible?

**Integration by parts**

13. Using integration by parts, determine the following integrals:

(a) $\displaystyle\int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx$, (b) $\displaystyle\int_0^\infty xe^{-2x}\,dx$, (c) $\displaystyle\int_0^1 x\log\left(\frac{1}{x}\right)\,dx$, (d) $\displaystyle\int_0^\infty x^3e^{-x^2}\,dx$.

14. By writing each of the following functions $f(x)$ in the form $1\cdot f(x)$, and using integration by parts, determine their indefinite integrals:

(a) $\log(x)$, (b) $\log^3(x)$, (c) $\cosh^{-1}(x)$, (d) $\tanh^{-1}(x)$, (e) $\sin(\log(x))$.

**Reduction formulae**

15. (a) Show that for $n \geq 1$, we have:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx + c,$$

where $c$ is an arbitrary constant. Hence, evaluate $\displaystyle\int \sin^6(x)\,dx$.

    (b) Using (a), show that the integral $I_n = \displaystyle\int_0^{\pi/2} \sin^n(x)\,dx$ satisfies $I_n = (n-1)I_n/n$. Hence, evaluate $I_2$ and $I_4$.

16. Establish reduction formulae for each of the following parametric integrals:

(a) $I_n = \int_0^\infty x^n e^{-x^2}\,dx$, (b) $J_n = \int_0^\pi x^{2n}\cos(x)\,dx$, (c) $K_n = \int_0^\infty x^{n-1}e^{-x}\,dx$, (d) $L_n = \int_0^\infty \frac{dx}{(1+x^2)^n}\,dx$.

Hence: (i) evaluate $I_3, I_5$; (ii) evaluate $J_3, J_5$; (iii) establish a general formula for $K_n$; (iv) evaluate $L_4$. $(*)$ Using part (c), suggest a reasonable definition of $z!$ where $z$ is a complex number. Will this work for all complex numbers?

### Miscellaneous integrals

[*This section contains a large collection of integrals from past papers for you to do. If you feel like you are getting too much of a good thing, feel free to save some of them for us to do together in the supervision.*]

17. Evaluate the following integrals, using the most efficient method in each case:

(a) $\int_4^9 \frac{dx}{\sqrt{x}-1}$

(b) $\int_{\pi/3}^{\pi/4} \frac{1+\tan^2(x)}{(1+\tan(x))^2}\,dx$

(c) $\int \frac{e^{2x}-2e^x}{e^{2x}+1}\,dx$

(d) $\int \frac{dx}{1+3\cos^2(x)}$

(e) $\int_2^3 \frac{2x+1}{x(x+1)}\,dx$

(f) $\int \frac{1}{2\sqrt{x}}e^{\sqrt{x}}\,dx$

(g) $\int x^3 e^{-x^4}\,dx$

(h) $\int \left(\frac{\sin(2x)}{\sin^2(x)+\log(x)} + \frac{1}{x(\sin^2(x)+\log(x))}\right)dx$

(i) $\int x\sqrt{3-2x}\,dx$

(j) $\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+6}\,dx$

(k) $\int \frac{\log(x)}{x^4}\,dx$

(l) $\int \sqrt{1-x^2}\,dx$

(m) $\int_{\pi/3}^{\pi/2} \tan(x)\cos^4(x)\,dx$

(n) $\int_1^5 x^2\log(x)\,dx$

(o) $\int e^x \sinh(3x)\,dx$

(p) $\int \frac{\arctan(x)}{x^2}\,dx$

(q) $\int_{e^3}^{e^4} \frac{3\log(x)-4}{x\log^2(x)-3x\log(x)+2x}\,dx$

(r) $\int_0^{\pi/6} x\sin(3x)\,dx$

(s) $\int \sin(2x)e^{\sin^2(x)}\,dx$

(t) $\int \frac{dx}{\cos^2(x)(\tan^3(x)-\tan(x))}$

(u) $\int_{-1/\pi}^{1/\pi} \sin^2(3x^3+2x)\log\left[\frac{1-x^5}{1+x^5}\right]dx$

(v) $\int \sin(2x)\cos(x)\,dx$

(w) $\int x\log(x)\,dx$

(x) $\int \frac{dx}{x\log(x)}$

(y) $\int \frac{\sinh^3(x)}{\cosh^2(x)}\,dx$

(z) $\int \frac{1}{\sin^2(3x+1)}\,dx$

**The fundamental theorem of calculus**

18. State both parts of the *fundamental theorem of calculus*. Use the fundamental theorem of calculus to evaluate the following derivatives:

$$\text{(a) } \frac{d}{dx}\int_1^x \frac{\log(t)\sin^2(t)}{t^2+7}\,dt, \quad \text{(b) } \frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right], \quad \text{(c) } \frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right].$$

19. Without evaluating the integrals, determine the local extrema of the functions $F_1, F_2$ defined by:

$$\text{(a) } F_1(x) = \int_0^x t^2\sin^2(t)\,dt, \qquad \text{(b) } F_2(x) = \int_{-\infty}^x e^{-t^2}\,dt.$$

Hence, sketch the graphs of the functions $F_1, F_2$. [*Note: $F_2(x) \to \sqrt{\pi}$ as $x \to \infty$; see Question 23!*]

**(†) Leibniz's integral rule**

20. Using the multivariable chain rule (we'll study it properly next term!), derive *Leibniz's integral rule*:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)\,dt.$$

Give a geometric explanation for the rule in terms of changing areas. Verify that the rule holds in the following cases:

(a) $a(x) = 0, b(x) = 1+x$, and $f(x,t) = t(x-t)$;
(b) $a(x) = \pi x^2, b(x) = x$, and $f(x,t) = 2x^2t + x\sin(t)$.

21. Evaluate the limit $\lim\limits_{x\to\infty} \dfrac{d}{dx}\displaystyle\int_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}\,dt.$

22. For all values of $x$, evaluate the integrals:

$$\text{(a) } f(x) = \int_0^1 \frac{t^x-1}{\log(t)}\,dt, \qquad \text{(b) } g(x) = \int_0^\infty \frac{\log(1+x^2t^2)}{1+t^2}\,dt, \qquad \text{(c) } h(x) = \int_0^1 \frac{\sin(x\log(t))}{\log(t)}\,dt,$$

by considering the derivatives $f'(x), g'(x), h'(x)$. This method is sometimes called *Feynman's trick* for integration.

23. This question determines the Gaussian integral in a different way to the lectures (you will use a transformation to polar coordinates on the next sheet!). Define:

$$f(x) = \left(\int_0^x e^{-t^2}\,dt\right)^2, \qquad \text{and} \qquad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{1+t^2}\,dt.$$

Show that $f'(x) + g'(x) = 0$, and hence deduce that $f(x) + g(x) = \pi/4$. Conclude that $\displaystyle\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$.

**(†) Integral inequalities**

24. Using a sketch, show that $\sin(x) \geq 2x/\pi$ for $0 \leq x \leq \pi/2$. Hence show that $\displaystyle\int_0^{\pi/2} \frac{x^2}{1+\sin^2(x)}\,dx < \frac{\pi^3}{8}\left(1-\frac{\pi}{4}\right)$.

25. State and prove *Schwarz's inequality* for integrals. Use it to show that $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{x^2+1}}\,dx < \sqrt{\frac{\pi}{4}\arctan\left(\frac{\pi}{2}\right)}$.