# Part IA: Mathematics for Natural Sciences B
# Examples Sheet 7: Multivariable integration and Gaussian integrals

*Please send all comments and corrections to jmm232@cam.ac.uk.*

Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions.

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**Area integrals in Cartesian coordinates**

1. Draw a convincing diagram showing the volume that is represented by the *area integral*:

$$\iint_D f(x,y)\,dA,$$

where $D$ is a region of the $xy$-plane, $f$ is a function of two variables, and $dA$ is a small area element in the $xy$-plane. What is the formula for $dA$ in Cartesian coordinates?

2. Evaluate the area integral:

$$\iint_D 2xy^2\,dA$$

over the triangle $D$ bounded by the lines $y = 1$, $y = \frac{1}{2}x$ and $x = 0$ by (a) integrating with respect to $y$ first, then with respect to $x$; (b) integrating with respect to $x$ first, then with respect to $y$.

3. Evaluate the following integrals, including a suitable sketch of the regions in each case:

(a) $\displaystyle\iint_{D_1} xe^{xy}\,dA$, (b) $\displaystyle\iint_{D_2} \sin(x+y)\,dA$, (c) $\displaystyle\iint_{D_3} x^2y\,dA$,

where:

- $D_1 = [0,\pi]\times[0,\pi]$,
- $D_2$ is the triangle bounded by the lines $y = 0, x = \frac{1}{2}\pi, y = x$,
- $D_3$ is the region contained in the first quadrant of the plane, bounded by the curves $y = 2, y = x^2$ and $y = 1/x$.

4. By considering the double integral:

$$I = \iint_D xe^{-xy}\,dA$$

over the region $D = \{(x,y) : x \geq 0, x/2 \leq y \leq x\}$, evaluate the integral:

$$\int_0^\infty \left(e^{-x^2} - e^{-2x^2}\right)\frac{dx}{x^2}.$$

5. ($*$) Show that:

$$\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dxdy \neq \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dydx.$$

Why can this happen here? What is the (signed) volume between $(x-y)/(x+y)^3$ and the plane $z = 0$?

### Area integrals in polar coordinates

6. With the aid of a diagram, derive the formula for the area element $dA$ in plane polar coordinates. Hence, evaluate the following integrals, including a suitable sketch of the regions in each case:

(a) $\displaystyle\iint_{D_1} x^2(1-x^2-y^2)\,dA,$ (b) $\displaystyle\iint_{D_2} \sqrt{x^2+y^2}\,dA,$

where:

- $D_1$ is the unit disk,
- $D_2$ is the region bounded by the curve $r = 2a(1+\cos(\theta))$ in plane polar coordinates, with $a$ constant.

7. Evaluate the double integral:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{0} e^{x^2+y^2}\,dydx.$$

8. A heap of building materials, of volume $V$, is stockpiled in a circle of radius $R$. Within the heap, $0 \le r \le R$, its height profile is given by $Re^{-r/R}$, where $r$ is radial distance from the centre of the heap. Determine $R$ in terms of $V$.

9. (a) Sketch the surfaces $x^2+y^2=16$ and $z=2x^2+2y^2$ in 3D. Using an appropriate double integral, determine the volume of the region bounded by these surfaces and the plane $z=0$.
   (b) Repeat part (a) with the second surface replaced by the surface $z^2=2x^2+2y^2$.

### Volume integrals in Cartesian coordinates

10. Evaluate the volume integral:

$$\iiint_V xyz^2\,dxdydz$$

where $V$ is:

   (a) the prism bounded by the planes $x=0, y=0, y=1, z=0$ and $x+z=1$;
   (b) the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

11. (a) Calculate the volume of the region defined by the constraints $|x| \le L$, $|y| \le b(2-z/H)$ and $H(x/L)^2 \le z \le H$, where $L, H, b$ are constants.
    (b) Calculate the volume of the intersection of this region with an infinite cylinder of diameter $D$, centred on the line $x=0, z=\frac{1}{2}D$. [*Hint: you don't need to do any integration here!*]

### Volume integrals in cylindrical and spherical polar coordinates

12. With the aid of diagrams, derive the formulae for the volume elements $dV$ in cylindrical polar coordinates and in spherical polar coordinates. Hence, using appropriate coordinates in each case, determine using multiple integration:

    (a) the volume of a cylinder with radius $a$ and height $h$;
    (b) the volume of a sphere with radius $a$;
    (c) the volume of a cone with *height* $a$, with sloped edge inclined at angle $\pi/4$ to the vertical;
    (d) the volume of a cone with *sloped edge* of length $a$, with sloped edge inclined at angle $\pi/4$ to the vertical, whose base is replaced by a sector of a sphere of radius $a$.

13. Find the volume bounded by the plane $z = 0$ and the surface:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z}{d} + 1,$$

where $a, d$ are positive constants.

14. (a) Find the volume of the cylindrically symmetric object bounded by the planes $z = 0$, $z = H$ and the surface of revolution $r = Re^{-z/H}$, where $R, H$ are constants.

    (b) Now suppose that the cross-section of the shape is altered so that the cylindrical symmetry is broken. Specifically, suppose that a sector of each cross-sectional disk is removed such that at $z = H$, half the disk is removed, but at $z = 0$, none of the disk is removed, and in-between the amount of area removed interpolates linearly between these two extremes. Calculate the volume of the resulting object.

15. (a) By considering scaled coordinates $x' = x/a$, $y' = y/b$ and $z' = z/c$, and by arguing how the volume element $dV$ changes in these coordinates, show that the volume bounded by the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

    is given by $4\pi abc/3$. Verify that this reduces to the expected result for a sphere.

    (b) Without detailed calculation, determine:

$$\iiint\limits_V (x^4y^2z + xy^2z^4)e^{-(x/a)^2-(y/b)^2-(z/c)^2}\,dxdydz,$$

    where $V$ is the interior of the ellipsoid in part (a).

**Mass and centre of mass**

16. Find the mass of a spherical body of radius $R$ whose mass density is:

$$\rho(r) = Ar^{-2}\left(1 - e^{-r/r_0}\right),$$

where $r$ is radial distance from the centre of the object, and $A, r_0$ are positive constants.

17. The centre of mass of an object occupying a volume $V$ is given by:

$$\bar{\mathbf{x}} = \frac{1}{M}\iiint\limits_V \rho(\mathbf{x})\mathbf{x}\,dV,$$

where $M$ is the mass of the object, and $\rho(\mathbf{x})$ is its mass density at the point $\mathbf{x}$.

    (a) Consider a wedge cut from a uniform sphere of radius $a$, defined by the spherical coordinates $0 \leq r \leq a$, $0 \leq \theta \leq \pi$, $-\beta/2 \leq \phi \leq \beta/2$. Show that the centre of mass of the wedge is a distance $af(\beta)$ from its straight edge, where:

$$f(\beta) = \frac{3\pi}{8\beta}\sin(\beta/2).$$

    Sketch the graph of $f(\beta)$ for $0 < \beta < 2\pi$, and comment on the physical relevance of your graph.

    (b) Compute the vector area of the curved surface of the wedge.

**Gaussian integrals**

18. By considering the integral of $e^{-(x^2+y^2)/a^2}$ over all of $\mathbb{R}^2$, prove that:

$$\int_{-\infty}^{\infty} e^{-x^2/a^2}\,dx = a\sqrt{\pi}.$$

Hence evaluate:

$$\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)/a^2}\,dV.$$

19. Evaluate:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2+b/x^2)}\,dx,$$

by making the substitution $y = \frac{1}{2}(\sqrt{a}x - \sqrt{b}/x)$. Hence or otherwise, evaluate:

$$\int_{-\infty}^{\infty} \frac{1}{x^2}e^{-\frac{1}{2}(ax^2+b/x^2)}\,dx.$$

20. Prove by induction that for non-negative integers $n$ and positive $a$:

$$\int_{0}^{\infty} x^{2n}e^{-ax^2}\,dx = \frac{(2n-1)(2n-3)...1}{2(2a)^n}\sqrt{\frac{\pi}{a}}.$$

Verify this result by repeatedly differentiating $\int_0^{\infty} e^{-ax^2}\,dx$ with respect to $a$.