

IA Mathematics for Natural Sciences A: Examples Sheet 2 - Vector products and triple products

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Vector products: basics

- (a) State two equivalent formulae for the vector product $\mathbf{v} \times \mathbf{w}$ of two vectors. Under what conditions are each of these formulae valid?
(b) Draw a diagram showing the vectors $\mathbf{v} = (1, 1, 0)^T$ and $\mathbf{w} = (1, 1, 1)^T$. Calculate $\mathbf{v} \times \mathbf{w}$ using both formulae you stated in part (a), and check they agree. Can we determine the angle between $\mathbf{v}, \mathbf{w}$ using the vector product?
- Using the vector product:
 - find an equation of the form $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ for the plane through $(1, 1, 1)$, $(1, 2, 3)$ and $(0, 0, 4)$;
 - determine whether the points $(0, 0, 2)$, $(0, 1, 3)$, $(1, 2, 3)$, $(2, 3, 4)$ are coplanar.
- Using the properties of the vector product, show that $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$.
- Explain why an equation for a line passing through the point $\mathbf{a}$, in the direction $\mathbf{v} \neq \mathbf{0}$, is $(\mathbf{r} - \mathbf{a}) \times \mathbf{v} = \mathbf{0}$. Hence, find Cartesian equations for the line through the points $(1, 0, 1)$, $(0, 2, 3)$.
- Derive a formula for the shortest distance between the lines $\mathbf{r}_1(\lambda) = \mathbf{a}_1 + \lambda \mathbf{v}_1$ and $\mathbf{r}_2(\mu) = \mathbf{a}_2 + \mu \mathbf{v}_2$. What happens if $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$? Find the shortest distance between the lines L_1, L_2 in Question 6.

Vector products: perpendicular projections

- Draw a diagram showing that the magnitude of the component of $\mathbf{v}$ perpendicular to $\mathbf{u} \neq \mathbf{0}$ is $|\mathbf{v} \times \hat{\mathbf{u}}|$. Show further that the projection of $\mathbf{v}$ perpendicular to $\mathbf{u}$ is $\hat{\mathbf{u}} \times (\mathbf{v} \times \hat{\mathbf{u}})$. Using these results:
 - Show that an equation of a line passing through the point $\mathbf{a}$, in the direction $\mathbf{v} \neq \mathbf{0}$, is $\mathbf{r} \times \mathbf{v} = \mathbf{m}$, where $|\mathbf{m}|/|\mathbf{v}|$ is the shortest distance to the origin. Find the equations of the lines L_1, L_2 from Sheet 1 Question 6 in the form $\mathbf{r} \times \mathbf{v} = \mathbf{m}$; hence, find their shortest distances to the origin.
 - Explain why an equation of the plane through $\mathbf{a}$ with normal $\mathbf{n} \neq \mathbf{0}$ is $\hat{\mathbf{n}} \times ((\mathbf{r} - \mathbf{a}) \times \hat{\mathbf{n}}) = \mathbf{r} - \mathbf{a}$.
- By considering appropriate perpendicular projections, show that:
 - the shortest distance from a point $\mathbf{p}$ to a line passing through a point $\mathbf{a}$ with direction vector $\mathbf{v}$ is given by $|(\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{v}}|$, and the closest point on the line to the point is $\mathbf{p} - \hat{\mathbf{v}} \times ((\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{v}})$.
 - the shortest distance from a point $\mathbf{p}$ to a plane passing through a point $\mathbf{a}$ with normal $\mathbf{n}$ is given by $\sqrt{|\mathbf{p} - \mathbf{a}|^2 - |(\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{n}}|^2}$.

Hence, use the vector product to find the shortest distance from the point $P = (1, 0, 1)$ to the planes P_1, P_2 and the lines L_1, L_2 from Sheet 1 Questions 6 and 7.

Vector and scalar triple products

8. State *Lagrange's formula* for the *vector triple product* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$; now, learn it off by heart. Give an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. Using Lagrange's formula:

(a) prove *Jacobi's identity*, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$;

(b) verify that the equation of the plane in Question 1(b) is equivalent to $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

9. Consider the equation $\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$, for the variable $\mathbf{r}$, where $\lambda \neq 0$ and $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$ are constant.

(a) By taking the scalar product of the equation with $\mathbf{a}$, show that $\lambda(\mathbf{r} \cdot \mathbf{a}) = \mathbf{c} \cdot \mathbf{a}$.

(b) By taking the vector product of the equation with $\mathbf{a}$, and using (a), find the equation's solution.

(c) Does the equation have any solutions when $\lambda = 0$ and $\mathbf{a} \cdot \mathbf{c} \neq 0$? What if $\lambda = 0$, and $\mathbf{a} \cdot \mathbf{c} = 0$? Find the solutions, if they exist, in each case. [Hint: what does (a) tell you about $\mathbf{r} \cdot \mathbf{a}$ in these cases?]

10. (a) Solve $2\mathbf{r} + \hat{\mathbf{n}} \times \mathbf{r} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r})^2 = \mathbf{a}$ for $\mathbf{r}$, where $\hat{\mathbf{n}}$ is a unit vector, and $\hat{\mathbf{n}} \cdot \mathbf{a} = -1$;

(b) Solve $\mathbf{r} = \mathbf{a} - (\mathbf{b} \cdot \mathbf{r})\mathbf{b} - \mathbf{b} \times \mathbf{r}$ for $\mathbf{r}$, where $\mathbf{a}, \mathbf{b}$ are constant vectors.

11. (a) Define the *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. With a diagram, show that the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ represents the *signed* volume of the parallelepiped formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(b) Hence, explain why: (i) the scalar triple product is antisymmetric on odd permutations, and symmetric on even permutations; (ii) three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a basis if and only if $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$.

(c) Let $\mathbf{a} = (0, \frac{1}{2}, \frac{1}{2})$, $\mathbf{b} = (\frac{1}{2}, 0, \frac{1}{2})$ and $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, 0)$. Compute $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$, $[\mathbf{b}, \mathbf{a}, \mathbf{c}]$ and $[\mathbf{b}, \mathbf{c}, \mathbf{a}]$, and comment on how they are related in light of (b)(i). Do the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a basis?

12. Simplify the scalar triple products $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

13. (a) Given the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are orthonormal, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Hence express $(2, 3, 4)$ in terms of the basis $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$.

(b) If instead the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are *not* orthonormal, derive expressions for the coefficients α, β, γ in the formula $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Hence express $(1, 1, 1)$ in terms of the basis $\{(1, 2, 1), (0, 0, 1), (2, -1, 1)\}$.

(c) Assuming $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a basis, we define the *reciprocal basis* to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be the set of vectors:

$$\mathbf{A} = (\mathbf{b} \times \mathbf{c}) / [\mathbf{a}, \mathbf{b}, \mathbf{c}], \quad \mathbf{B} = (\mathbf{c} \times \mathbf{a}) / [\mathbf{a}, \mathbf{b}, \mathbf{c}], \quad \mathbf{C} = (\mathbf{a} \times \mathbf{b}) / [\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

Show that $\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$, and $\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{c} = \mathbf{C} \cdot \mathbf{a} = \mathbf{C} \cdot \mathbf{b} = 0$. Find the reciprocal basis to the basis in part (b), and explain how to use it to find the coefficients α, β, γ .