

IA Mathematics for Natural Sciences A: Examples Sheet 5 - Derivatives, and integration basics

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Questions marked with a (*) are difficult and should not be attempted at the expense of the other questions.

Derivatives and curve-sketching

1. (a) From the limit definition, find the derivative of x^n , where n is a non-negative integer.
(b) From the limit definition, show that the derivative of a^x is proportional to a^x . [We define e to be the value of a for which the proportionality constant is one.]
(c) State the *product rule* for derivatives, and derive it directly from the limit definition.
2. State the chain rule, the quotient rule, and the reciprocal rule for derivatives. [These can similarly be derived directly from the limit definition.] Using appropriate rules in each case, find the derivatives of:

$$(a) \log(x), \quad (b) 3^{x^2}, \quad (c) e^x/(x^3 - 1), \quad (d) x^3 \log(x^2 - 7), \quad (e) \sqrt{x^3 - e^x \log(x)}.$$

3. By writing each of the following functions in terms of exponentials, compute their derivatives: (a) $\cos(x)$; (b) $\sin(x)$; (c) $\cosh(x)$; (d) $\sinh(x)$; (e) $\tan(x)$; (f) $\tanh(x)$. Learn these derivatives off by heart.
4. Using the reciprocal rule, find the derivatives of: (a) $\cos^{-1}(x)$; (b) $\sin^{-1}(x)$; (c) $\cosh^{-1}(x)$; (d) $\sinh^{-1}(x)$; (e) $\tan^{-1}(x)$; (f) $\tanh^{-1}(x)$. Learn these derivatives off by heart. Now, by first expressing $\sin^{-1}(x)$ and $\sinh^{-1}(x)$ as logarithms instead, then differentiating, verify your results for (b) and (d).
5. Let $y = y(x)$ be a function of x . Using the reciprocal rule and the chain rule, show that:

$$\frac{d^3 x}{dy^3} = - \left(\frac{dy}{dx} \right)^{-4} \frac{d^3 y}{dx^3} + 3 \left(\frac{dy}{dx} \right)^{-5} \left(\frac{d^2 y}{dx^2} \right)^2,$$

and verify this result when $y = e^{2x}$.

6. Explain what is meant by *implicit differentiation*, and why it is called *implicit*. Using: (a) implicit differentiation; (b) the reciprocal rule, find dy/dx given $y + e^y \sin(y) = 1/x$.
7. State what it means for a function to be *even* or *odd*, and the graphical significance of these definitions. Decide whether the following functions are even, odd, both, or neither:

$$(a) x, \quad (b) \sin(x), \quad (c) e^x, \quad (d) \sin\left(\frac{\pi}{2} - x\right), \quad (e) |x| \cos(x), \quad (f) \sqrt{x}, \quad (g) 2, \quad (h) 0, \quad (i) \log \left| \frac{1+x}{1-x} \right|.$$

8. Give a list of things one should consider when sketching the graph of a single-variable function, $y = f(x)$. Sketch the graphs of the following functions, justifying your sketch in each case:

$$(a) (x-3)^3 + 2x, \quad (b) \frac{x}{1+x^2}, \quad (c) \frac{x^2+3}{x-1}, \quad (d) xe^x, \quad (e) \frac{\log(x)}{1+x}, \quad (f) \frac{1}{1-e^x}, \quad (g) e^x \cos(x).$$

Riemann sums

9. Explain what is meant by a *Riemann sum* for a function $f : [a, b] \rightarrow \mathbb{R}$ using a *partition* $P = (x_0, \dots, x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, \dots, t_n)$. How can we use a sequence of Riemann sums to find the value of an integral from 'first principles'? Using appropriate sequences of Riemann sums in each case, find the integrals:

$$(a) \int_0^1 x \, dx, \quad (b) \int_0^1 x^2 \, dx, \quad (c) \int_0^1 \sqrt{x} \, dx, \quad (d) \int_0^1 \cos(x) \, dx, \quad (e) \int_1^\infty \frac{dx}{x^{1+\alpha}}, \text{ for } \alpha > 0.$$

10. (*) If a sequence of Riemann sums for a function $f : [a, b] \rightarrow \mathbb{R}$ converges, must the function f be integrable? Give a proof or counterexample.

The fundamental theorem(s) of calculus

11. By considering a small change in x , give a graphical proof of the *first fundamental theorem of calculus*:

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x),$$

for constant a . Why does the right hand side not depend on a ? Using the theorem, evaluate:

$$(a) \frac{d}{dx} \int_1^x \frac{\log(t) \sin^2(t)}{t^2 + 7} \, dt, \quad (b) \frac{d}{dx} \left[\sum_{n=0}^N \binom{N}{n} \int_n^x \sin(y^2 + y^6) \, dy \right], \quad (c) \frac{d}{dx} \left[\sin(x) \int_x^0 \sin(\cos(t)) \, dt \right].$$

12. Without evaluating the integrals, determine the local extrema of the functions F_1, F_2 defined by:

$$(a) F_1(x) = \int_0^x t^2 \sin^2(t) \, dt, \quad (b) F_2(x) = \int_{-\infty}^x e^{-t^2} \, dt.$$

Hence, sketch the graphs of F_1, F_2 . [Assume that $F_2(x) \rightarrow \sqrt{\pi}$ as $x \rightarrow \infty$; we will prove this later!]

13. Explain what is meant by an *indefinite integral* $F(x)$ of the function $f(x)$. Using the first fundamental theorem of calculus, deduce that if $F(x)$ is any indefinite integral of $f(x)$, we have:

$$\int_a^x f(t) \, dt = F(x) - F(a),$$

which is the *second fundamental theorem of calculus*. Explain how we can use the theorem to find definite integrals without resorting to Riemann sums.