

IA Mathematics for Natural Sciences A: Examples Sheet 6 - Methods of integration

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Standard integrals

1. Using your knowledge of derivatives, *write down* the indefinite integrals of each of the following functions, where $a \neq 0$, $\alpha \neq -1$ and f is any (differentiable, non-zero) function:

(a) $(ax + b)^\alpha$	(b) e^{ax+b}	(c) $(ax + b)^{-1}$	(d) $\sin(ax + b)$
(e) $\cos(ax + b)$	(f) $\sec^2(ax + b)$	(g) $\operatorname{cosec}^2(ax + b)$	(h) $\sinh(ax + b)$
(i) $\cosh(ax + b)$	(j) $f'(x)f(x)^\alpha$	(k) $\frac{f'(x)}{f(x)}$	(l) $\frac{1}{\sqrt{1 - (ax + b)^2}}$
(m) $\frac{1}{\sqrt{(ax + b)^2 - 1}}$	(n) $\frac{1}{\sqrt{1 + (ax + b)^2}}$	(o) $\frac{1}{1 + (ax + b)^2}$	(p) $\frac{1}{1 - (ax + b)^2}$

Learn all of these integrals off by heart, and get your supervision partner to test you on them.

2. Using the results of the previous question, and the second fundamental theorem of calculus, evaluate the definite integrals:

(a) $\int_0^2 (x - 1)^2 dx$	(b) $\int_0^\pi e^{i\theta} d\theta$	(c) $\int_0^\pi \cos(x) dx$
(d) $\int_{-\pi/4}^{\pi/4} \sec^2(x) dx$	(e) $\int_0^1 \frac{2x + 4}{x^2 + 4x + 1} dx$	(f) $\int_0^1 \sinh(2x) \cosh^3(2x) dx$
(g) $\int_0^\infty \frac{dx}{9 + x^2}$	(h) $\int_0^1 \frac{dx}{\sqrt{x^2 + x + 1}}$	(i) $\int_0^3 \frac{8 - 2x}{\sqrt{6x - x^2}} dx$

3. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax} \cos(bx)$. Similarly, find the indefinite integrals of $e^x(\sin(x) - \cos(x))$ and $e^x(\sin(x) + \cos(x))$.
4. (*Rational functions*) Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

(a) $\frac{1}{1 - x^2},$	(b) $\frac{3x}{2x^2 + x - 1},$	(c) $\frac{x^4 + x^2 + 4x + 6}{3 + 2x - 2x^2 - 2x^3 - x^4}.$
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Compare your answer to part (a) with your answer to Question 1(p). Are your results compatible?

Integration by substitution

5. Using substitutions, find: (a) $\int \frac{dx}{\sqrt{1 - x^2}}$; (b) $\int \frac{dx}{\sqrt{1 + 9x^2}}$, verifying agreement with Question 1.

6. By means of appropriate substitutions in each case, or otherwise, determine the indefinite integrals of the following functions:

$$(a) x\sqrt{x+3}, \quad (b) \tan(x)\sqrt{\sec(x)}, \quad (c) \frac{e^x}{\sqrt{1-e^{2x}}}, \quad (d) \frac{1}{x\sqrt{x^2-1}}.$$

7. The *Weierstrass t-substitution* sets $t = \tan(\frac{1}{2}x)$. Show that under this substitution, $\sin(x) = 2t/(1+t^2)$, $\cos(x) = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$. Hence, find the indefinite integrals of:

$$(a) \operatorname{cosec}(x), \quad (b) \sec(x), \quad (c) \frac{1}{2+\cos(x)}.$$

8. (*A subtle substitution*) For any sufficiently well-behaved function $f(x)$, prove the *Cauchy-Schlömilch formula*:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx,$$

by setting $u = x - 1/x$. [Hint: sketch u as a function of x .] Hence, show that $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$.

Integration by parts

9. State and prove the *integration by parts* formula. Using integration by parts, determine the following integrals:

$$(a) \int_{-\pi/2}^{\pi/2} x \sin(2x) dx, \quad (b) \int_0^{\infty} x e^{-2x} dx, \quad (c) \int_0^1 x \log\left(\frac{1}{x}\right) dx, \quad (d) \int_0^{\infty} x^3 e^{-x^2} dx.$$

10. By writing each of the following functions $f(x)$ in the form $1 \cdot f(x)$, and using integration by parts, determine their indefinite integrals:

$$(a) \log(x), \quad (b) \log^3(x), \quad (c) \cosh^{-1}(x), \quad (d) \tanh^{-1}(x), \quad (e) \sin(\log(x)).$$

Reduction formulae

11. (a) Show that for $n \geq 1$, we have: $\int \sin^n(x) dx = -\frac{1}{n} \cos(x) \sin^{n-1}(x) + \frac{n-1}{n} \int \sin^{n-2}(x) dx$.

Hence, evaluate $\int \sin^6(x) dx$.

- (b) Using part (a), show that $I_n = \int_0^{\pi/2} \sin^n(x) dx$ satisfies $I_n = (n-1)I_n/n$. Hence find I_2, I_4 .

12. Establish reduction formulae for each of the following parametric integrals:

$$(a) I_n = \int_0^{\infty} x^n e^{-x^2} dx, \quad (b) J_n = \int_0^{\pi} x^{2n} \cos(x) dx, \quad (c) K_n = \int_0^{\infty} x^{n-1} e^{-x} dx, \quad (d) L_n = \int_0^{\infty} \frac{dx}{(1+x^2)^n}.$$

Using your reduction formulae, find I_3, I_5, J_3, J_5 and L_4 . Establish also a general formula for K_n .