IA Mathematics for Natural Sciences B:
Examples Sheet 5 - Infinite series, and Taylor series

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Questions marked with a (∗) are difficult and should not be attempted at the expense of the other questions. A section marked with a (†) contains content that is unique to the Mathematics B course.

(†) Convergence of infinite series

1. (a) What does it mean to say that the infinite series $u_1 + u_2 + \cdots$ *converges*? What does it mean to say that the infinite series $u_1 + u_2 + \cdots$ *converges absolutely*? How are these two conditions related?

   (b) By directly evaluating the relevant partial sums, decide whether the following series:

   $$\sum_{n=1}^{\infty} ar^n, \qquad \sum_{n=1}^{\infty} anr^n, \qquad \sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right),$$

   where a, r are constant with $|r| < 1$, are (absolutely) convergent.

2. (a) State the *comparison test*, and use it to show the harmonic series diverges. Hence, show that the following definition of the Riemann zeta function:

   $$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p},$$

   with p real, converges if and only if $p > 1$. For what values of p is it absolutely convergent?

   (b) State, and prove, the *alternating series test* for series convergence. Hence, show that the following definition of the Dirichlet eta function:

   $$\eta(p) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p},$$

   with p real, converges if and only if $p > 0$. For what values of p is it absolutely convergent?

3. State the *ratio test*, and use it to show $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges absolutely. To what does it converge?

4. (*Miscellaneous series*) Using an appropriate test in each case, determine whether the following series are (i) convergent; (ii) absolutely convergent:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+4}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{n!}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{5 \cdot 8 \cdot 11 \cdot \ldots \cdot (3n+2)}{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4n-3)}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

   (g) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2n+5}{3n+1}\right)^n$ (h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log(n)}$ (i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \log(n)}$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$ (k) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$ (l) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - n^2\right)$

Taylor series

5. Clearly state Taylor's theorem, giving Lagrange's formula for the remainder term. Hence, obtain the first three non-zero terms in the Taylor series of $\log(x)$ about $x = 1$ by direct differentiation. Using this expansion, together with Lagrange's form of the remainder, show that:

$$|\log(3/2) - 5/12| \leq 1/64,$$

and hence give an approximation of $\log(3/2)$ valid to one decimal place.

6. Write down the Taylor series about $x = 0$ for the following functions, finding their range of convergence by appropriate tests in each case:

$$\text{(a) } e^x, \quad \text{(b) } \log(1+x), \quad \text{(c) } \sin(x), \quad \text{(d) } \cos(x), \quad \text{(e) } \sinh(x), \quad \text{(f) } \cosh(x), \quad \text{(g) } (1+x)^a,$$

where in the final part $a \in \mathbb{R}$ is any real number (ignore endpoints of the range of convergence in the final case, where convergence is subtle). What happens when a is a non-negative integer? Learn these series off by heart, and get your supervision partner to test you on them.

7. (*Miscellaneous Taylor series*) Using the most efficient method in each case (this rarely involves differentiating!), find the first three non-zero terms in the Taylor expansions of the following functions:

(a) $\dfrac{1}{\sqrt{1+x}}$ about $x = 0$;	(b) $\dfrac{1}{(x^2+2)^{3/2}}$ about $x = 0$;	(c) $x\sinh(x^2)$ about $x = 0$;
(d) $\tan(x)$ about $x = 0$;	(e) $\log(\cos(x))$ about $x = 0$;	(f) $(\cosh(x))^{-1/2}$ about $x = 0$;
(g) $\log(2+x)$ about $x = 0$;	(h) $\sin^6(x)$ about $x = 0$;	(i) $\arcsin(x)$ about $x = 0$;
(j) $\dfrac{\sinh(x+1)}{x+2}$ about $x = -1$;	(k) $e^{\sin(x)}$ about $x = \pi/2$;	(l) $\cosh(\log(x))$ about $x = 2$.

8. Find the value of the thirty-second derivative of $\cos(x^4)$ at $x = 0$.

9. Let $f(x)$ be a function which can be expanded as a Taylor series near $x = 0$. Find the first two non-zero terms in the Taylor series of $\log(1 + f(x))$ about $x = 0$, assuming $1 + f(0) > 0$, $f'(0) \neq 0$, and $f''(0)(1 + f(0)) \neq (f'(0))^2$. Why do we need these conditions?

10. Find the first three non-zero terms in a series approximation of $\log(1 + x + 2x^2) - \log(x^2)$, as $x \to \infty$.

11. (a) Using the Taylor series for $\log(1 + x)$, show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots = \log(2)$.

    (b) $(*)$ Hence, by an appropriate sequence of transformations of the series in part (a), show that:

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots = \tfrac{3}{2}\log(2).$$

    Comment on this result. [*Look up the Riemann rearrangement theorem!*]

12. $(*)$ Sketch the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \exp(-1/x^2)$ for $x \neq 0$, and $f(0) = 0$. Find its Taylor series about $x = 0$. Comment on the general utility of Taylor series.

(†) Landau notation

13. Give the formal definition of *Landau's big-O notation*, '$f(x) = O(g(x))$ as $x \to x_0$'. Using the formal definition, decide which of the following statements are true.

$$\text{(a) } x = O(x^2) \text{ as } x \to 0, \quad \text{(b) } x^2 = O(x) \text{ as } x \to 0, \quad \text{(c) } x = O(x^2) \text{ as } x \to \infty, \quad \text{(d) } x^2 = O(x) \text{ as } x \to \infty.$$

14. Give the leading terms in an approximation to each of the following functions in the given limits, indicating the leading behaviour of the remainder in Landau's big-O notation:

$$\text{(a) } \frac{x^3+x}{x+2} \text{ as } x \to 0, \qquad \text{(b) } \frac{\cos(x)-1}{x^3} \text{ as } x \to 0, \qquad \text{(c) } \frac{1+2x+2x^2}{3x+3} \text{ as } x \to \infty.$$

15. Show that $(x^3+x^2+1)^{1/3} - (x^2+x)^{1/2} = -\frac{1}{6}x + \frac{1}{72x} + O\left(\frac{1}{x^2}\right)$ as $x \to \infty$.

Newton-Raphson iteration

16. Give an explanation of the Newton-Raphson algorithm for root finding, including an appropriate sketch. Under what general conditions is it guaranteed that Newton-Raphson will converge to the root of interest? Prove that, when it converges to the root of interest, the Newton-Raphson method enjoys *quadratic convergence*.

17. (a) Find the value of the first iterate of Newton-Raphson iteration for the function $f(x) = x - 2 + \log(x)$ with a starting guess of $x_0 = 1$.

    (b) Find the value of the first and second iterates of Newton-Raphson iteration, valid to two decimal places, for the function $f(x) = x^2 - 2$ with a starting guess of $x_0 = 1$.

18. Let $f(x) = x^2 - 2\epsilon x - 1$, where ϵ is a small positive parameter ($0 < \epsilon \ll 1$). Let x_i be the ith Newton-Raphson iterate, with a starting guess of $x_0 = 1$, and let x_* be the unique positive root satisfying $f(x_*) = 0$. By Taylor expansion, show that $|x_i - x_*| \propto \epsilon^{n_i}$, where: (a) $n_0 = 1$; (b) $n_1 = 2$; (c) $n_2 = 4$.

19. ($*$) Consider the cubic equation $x^3 - 2x + 2 = 0$. Perform a numerical investigation (for example, by writing some simple code) to determine the ranges of starting guesses x_0 for Newton-Raphson which converge to the various roots, if any. Comment on the sensitivity of Newton-Raphson to the choice of initial guess x_0. [*Afterwards, look up Newton fractals - it is particularly interesting to see the behaviour of Newton-Raphson in the complex plane!*]