IA Mathematics for Natural Sciences B:
Examples Sheet 6 - Single-variable integration

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Questions marked with a (∗) are difficult and should not be attempted at the expense of the other questions. A section marked with a (†) contains content that is unique to the Mathematics B course.

Riemann sums

1. Explain what is meant by a *Riemann sum* for a function $f : [a, b] \to \mathbb{R}$ using a *partition* $P = (x_0, ..., x_n)$ (with $x_0 = a, x_n = b$) and *tagging* $T = (t_1, ..., t_n)$. How can we use a sequence of Riemann sums to find the value of an integral from 'first principles'? Using appropriate sequences of Riemann sums in each case, find the integrals:

   (a) $\int_0^1 x\,dx$, (b) $\int_0^1 x^2\,dx$, (c) $\int_0^1 \sqrt{x}\,dx$, (d) $\int_0^1 \cos(x)\,dx$, (e) $\int_1^\infty \frac{dx}{x^{1+\alpha}}$, for $\alpha > 0$.

2. (∗) If a sequence of Riemann sums for a function $f : [a, b] \to \mathbb{R}$ converges, must the function f be integrable? Give a proof or counterexample.

The fundamental theorem(s) of calculus

3. By considering a small change in x, give a graphical proof of the *first fundamental theorem of calculus*:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

   for constant a. Why does the right hand side not depend on a? Using the theorem, evaluate:

   (a) $\frac{d}{dx}\int_1^x \frac{\log(t)\sin^2(t)}{t^2+7}\,dt$, (b) $\frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2+y^6)\,dy\right]$, (c) $\frac{d}{dx}\left[\sin(x)\int_x^0 \sin(\cos(t))\,dt\right]$.

4. Without evaluating the integrals, determine the local extrema of the functions F_1, F_2 defined by:

   (a) $F_1(x) = \int_0^x t^2\sin^2(t)\,dt$, (b) $F_2(x) = \int_{-\infty}^x e^{-t^2}\,dt$.

   Hence, sketch the graphs of F_1, F_2. [*Assume that* $F_2(x) \to \sqrt{\pi}$ *as* $x \to \infty$*; we will prove this later!*]

5. Explain what is meant by an *indefinite integral* $F(x)$ of the function $f(x)$. Using the first fundamental theorem of calculus, deduce that if $F(x)$ is *any* indefinite integral of $f(x)$, we have:

$$\int_a^x f(t)\,dt = F(x) - F(a),$$

   which is the *second fundamental theorem of calculus*. Explain how we can use the theorem to find definite integrals without resorting to Riemann sums.

Standard integrals

6. Using your knowledge of derivatives, *write down* the indefinite integrals of each of the following functions, where $a \neq 0$, $\alpha \neq -1$ and f is any (differentiable, non-zero) function:

   (a) $(ax+b)^\alpha$ (b) e^{ax+b} (c) $(ax+b)^{-1}$ (d) $\sin(ax+b)$

   (e) $\cos(ax+b)$ (f) $\sec^2(ax+b)$ (g) $\operatorname{cosec}^2(ax+b)$ (h) $\sinh(ax+b)$

   (i) $\cosh(ax+b)$ (j) $f'(x)f(x)^\alpha$ (k) $\dfrac{f'(x)}{f(x)}$ (l) $\dfrac{1}{\sqrt{1-(ax+b)^2}}$

   (m) $\dfrac{1}{\sqrt{(ax+b)^2-1}}$ (n) $\dfrac{1}{\sqrt{1+(ax+b)^2}}$ (o) $\dfrac{1}{1+(ax+b)^2}$ (p) $\dfrac{1}{1-(ax+b)^2}$

   Learn all of these integrals off by heart, and get your supervision partner to test you on them.

7. Using the results of the previous question, evaluate the definite integrals:

   (a) $\displaystyle\int_0^2 (x-1)^2\,dx$ (b) $\displaystyle\int_0^\pi e^{i\theta}\,d\theta$ (c) $\displaystyle\int_0^\pi \cos(x)\,dx$

   (d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$ (e) $\displaystyle\int_0^1 \frac{2x+4}{x^2+4x+1}\,dx$ (f) $\displaystyle\int_0^1 \sinh(2x)\cosh^3(2x)\,dx$

   (g) $\displaystyle\int_0^\infty \frac{dx}{9+x^2}$ (h) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2+x+1}}$ (i) $\displaystyle\int_0^3 \frac{8-2x}{\sqrt{6x-x^2}}\,dx$

8. By writing $\cos(bx)$ as the real part of a complex exponential, determine the indefinite integral of $e^{ax}\cos(bx)$. Similarly, find the indefinite integrals of $e^x(\sin(x)-\cos(x))$ and $e^x(\sin(x)+\cos(x))$.

9. (*Rational functions*) Explain the general strategy that one should adopt when integrating a rational function. Hence, determine the indefinite integrals of the following rational functions by decomposing into partial fractions:

   (a) $\dfrac{1}{1-x^2}$, (b) $\dfrac{3x}{2x^2+x-1}$, (c) $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}$.

   Compare your answer to part (a) with your answer to Question 6(p). Are your results compatible?

Integration by substitution

10. Using substitutions, find: (a) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$; (b) $\displaystyle\int \frac{dx}{\sqrt{1+9x^2}}$, verifying agreement with Question 6.

11. By means of appropriate substitutions in each case, or otherwise, determine the indefinite integrals of the following functions:

    (a) $x\sqrt{x+3}$, (b) $\tan(x)\sqrt{\sec(x)}$, (c) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$, (d) $\dfrac{1}{x\sqrt{x^2-1}}$.

12. The *Weierstrass t-substitution* sets $t = \tan(\frac{1}{2}x)$. Show that under this substitution, $\sin(x) = 2t/(1+t^2)$, $\cos(x) = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$. Hence, find the indefinite integrals of:

    (a) $\operatorname{cosec}(x)$, (b) $\sec(x)$, (c) $\dfrac{1}{2+\cos(x)}$.

13. (*A subtle substitution*) For any sufficiently well-behaved function $f(x)$, prove the *Cauchy-Schlömlich formula*:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right)\,dx,$$

by setting $u = x - 1/x$. [*Hint: sketch u as a function of x.*] Hence, show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}$.

Integration by parts

14. State and prove the *integration by parts* formula. Using integration by parts, determine the following integrals:

$$\text{(a)} \int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx, \qquad \text{(b)} \int_0^{\infty} xe^{-2x}\,dx, \qquad \text{(c)} \int_0^1 x\log\left(\frac{1}{x}\right)\,dx, \qquad \text{(d)} \int_0^{\infty} x^3e^{-x^2}\,dx.$$

15. By writing each of the following functions $f(x)$ in the form $1 \cdot f(x)$, and using integration by parts, determine their indefinite integrals:

(a) $\log(x)$, (b) $\log^3(x)$, (c) $\cosh^{-1}(x)$, (d) $\tanh^{-1}(x)$, (e) $\sin(\log(x))$.

Reduction formulae

16. (a) Show that for $n \geq 1$, we have: $\displaystyle\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$. Hence, evaluate $\displaystyle\int \sin^6(x)\,dx$.

(b) Using part (a), show that $I_n = \displaystyle\int_0^{\pi/2} \sin^n(x)\,dx$ satisfies $I_n = (n-1)I_n/n$. Hence find I_2, I_4.

17. Establish reduction formulae for each of the following parametric integrals:

$$\text{(a) } I_n = \int_0^{\infty} x^n e^{-x^2}\,dx, \quad \text{(b) } J_n = \int_0^{\pi} x^{2n}\cos(x)\,dx, \quad \text{(c) } K_n = \int_0^{\infty} x^{n-1}e^{-x}\,dx, \quad \text{(d) } L_n = \int_0^{\infty} \frac{dx}{(1+x^2)^n}\,dx.$$

Using your reduction formulae, find I_3, I_5, J_3, J_5 and L_4. Establish also a general formula for K_n.

(†) The Leibniz integral rule

18. State Leibniz's integral rule for $\displaystyle\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt$, and give a graphical proof. Verify the rule when:

(a) $a(x) = 0$, $b(x) = 1 + x$, and $f(x,t) = t(x-t)$;
(b) $a(x) = \pi x^2$, $b(x) = x$, and $f(x,t) = 2x^2t + x\sin(t)$.

19. Evaluate the limit: $\displaystyle\lim_{x\to\infty} \frac{d}{dx}\int_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}\,dt$.

20. (*Feynman's trick*) For all values of x, evaluate the integrals:

$$\text{(a) } f(x) = \int_0^1 \frac{t^x - 1}{\log(t)}\, dt, \qquad \text{(b) } g(x) = \int_0^\infty \frac{\log(1 + x^2 t^2)}{1 + t^2}\, dt, \qquad \text{(c) } h(x) = \int_0^1 \frac{\sin(x \log(t))}{\log(t)}\, dt.$$

(†) Integral inequalities

21. Using a sketch, show that $\sin(x) \geq 2x/\pi$ for $x \in [0, \frac{\pi}{2}]$. Hence prove: $\displaystyle\int_0^{\pi/2} \frac{x^2}{1 + \sin^2(x)}\, dx \leq \frac{\pi^2}{8}\left(1 - \frac{\pi}{4}\right)$.

22. State and prove *Schwarz's inequality* for integrals. Use it to show that: $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{x^2 + 1}} \leq \sqrt{\frac{\pi}{4} \arctan\left(\frac{\pi}{2}\right)}$.