IA Mathematics for Natural Sciences B:
Examples Sheet 7 - Multivariable integration, and Gaussian integrals

Please send all comments and corrections to jmm232@cam.ac.uk.

Area integrals

1. Draw a convincing diagram giving the geometric interpretation of the *area integral* $\int_D f(x,y)\,dA$, where D is a region of the xy-plane, f is a function of two variables, and dA is a small area element in the xy-plane. Derive formulae for dA in: (a) Cartesian coordinates; (b) plane polar coordinates.

2. Evaluate the area integral $\int_D 2xy^2\,dA$ where D is the triangle bounded by $y = 1, y = \frac{1}{2}x$ and $x = 0$, by: (a) integrating with respect to y first, then x; (b) integrating with respect to x first, then y.

3. Including a sketch of the region of integration in each case, evaluate the following integrals:

$$\text{(a)} \int_{D_1} xe^{xy}\,dA, \qquad \text{(b)} \int_{D_2} \sin(x+y)\,dA, \qquad \text{(c)} \int_{D_3} x^2 y\,dA,$$

   where $D_1 = [0,\pi] \times [0,\pi]$, D_2 is the triangle bounded by the lines $y = 0, x = \frac{\pi}{2}, y = x$, and D_3 is the region contained in the first quadrant of the plane bounded by the curves $y = 2, y = x^2$ and $y = 1/x$.

4. Including a sketch of the region of integration in each case, evaluate the integrals:

$$\text{(a)} \int_{D_1} x^2(1 - x^2 - y^2)\,dA, \qquad \text{(b)} \int_{D_2} \sqrt{x^2+y^2}\,dA,$$

   where D_1 is the unit disk, and D_2 is the region bounded by the curve $r = 2a(1+\cos(\theta))$ in plane polar coordinates, with $a > 0$ constant.

5. Evaluate the double integral: $\int_0^3 \int_{-\sqrt{9-x^2}}^{0} e^{x^2+y^2}\,dydx$.

6. A heap of building materials, of volume V, is stockpiled in a circle of radius R. Within the heap, the height profile is $Re^{-r/R}$, where r is radial distance from the centre. Determine R in terms of V.

7. Find the volume bounded by $z = 0, x^2 + y^2 = 16$ and $z = 2x^2 + 2y^2$. Now, repeat this exercise with $z = 2x^2 + 2y^2$ replaced by $z^2 = 2z^2 + 2y^2$.

Volume integrals

8. Evaluate the volume integral $\int_V xyz^2\,dV$ where V is:

   (a) the prism bounded by the planes $x = 0, y = 0, y = 1, z = 0$ and $x + z = 1$;

   (b) the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

9. Calculate the volume of the region defined by the constraints $|x| \leq L$, $|y| \leq b(2 - z/H)$ and $H(x/L)^2 \leq z \leq H$, where L, H, b are constants.

10. Using appropriate diagrams, derive the formulae for the volume element dV in cylindrical polar coordinates and in spherical polar coordinates. Hence, determine using multiple integration:
    (a) the volume of a cylinder with radius a and height h;
    (b) the volume of a sphere with radius a;
    (c) the volume of a cone with *height* a, with sloped edge inclined at angle $\pi/4$ to the vertical;
    (d) the volume of a cone with *sloped edge* of length a, with sloped edge inclined at angle $\pi/4$ to the vertical, whose base is replaced by a sector of a sphere of radius a centred on the cone's apex.

11. Find the volume bounded by $z = 0$ and $(x^2 + y^2)/a^2 = z/d + 1$, where $a, d > 0$ are constant.

12. (a) Find the volume bounded by $z = 0, z = H$ and the surface of revolution $r = Re^{-z/H}$, where R, H are constant.
    (b) Suppose now that the cross-section of the shape in part (a) is altered so that the cylindrical symmetry is broken. Specifically, suppose that sector of each cross-sectional disk is removed such that at $z = H$, half the disk is removed, but at $z = 0$, none of the disk is removed, and in-between the amount of area removed interpolates linearly between these two extremes. Calculate the volume of the resulting object.

13. (a) By considering scaled coordinates $x' = x/a$, $y' = y/b$, $z' = z/c$, show that the volume V bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $4\pi abc/3$.
    (b) Without detailed calculation, evaluate: $\int_V (x^4y^2z + xy^2z^4)e^{-(x/a)^2-(y/b)^2-(z/c)^2} dV$.

14. A spherical body of radius R has mass density $\rho(r) = Ar^{-2}(1 - e^{-r/r_0})$, where r is radial distance from the centre of the object, and $A, r_0 > 0$ are constant. Find its mass.

15. The *centre of mass* of an object occupying a volume V is $\bar{\mathbf{x}} = \frac{1}{M}\int_V \rho(\mathbf{x})\mathbf{x}\, dV$, where M is its mass and $\rho(\mathbf{x})$ is its mass density at $\mathbf{x}$. Find the centre of mass of a uniform spherical wedge of radius a, defined by the coordinates $0 \leq r \leq a$, $0 \leq \theta \leq \pi$, $-\beta/2 \leq \phi \leq \beta/2$.

Gaussian integrals

16. By integrating $e^{-(x^2+y^2)/a^2}$ over all of $\mathbb{R}^2$, prove that $\int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx = a\sqrt{\pi}$. Hence evaluate: $\int_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)/a^2}\, dV$.

17. Using induction, show that:
$$\int_0^{\infty} x^{2n} e^{-ax^2}\, dx = \frac{(2n-1)(2n-3)...1}{2(2a)^n}\sqrt{\frac{\pi}{a}}.$$
    Now, prove this result by instead differentiating repeatedly with respect to a.

18. By considering $\int_D xe^{-xy}\, dA$ where $D = \{(x, y) : x \geq 0, x/2 \leq y \leq x\}$, evaluate: $\int_0^{\infty} \left(e^{-x^2} - e^{-2x^2}\right)\frac{dx}{x^2}$.

19. Using the substitution $y = \frac{1}{2}(\sqrt{a}x - \sqrt{b}/x)$, evaluate $\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2+b/x^2)}\, dx$. Hence find $\int_{-\infty}^{\infty} \frac{1}{x^2} e^{-\frac{1}{2}(ax^2+b/x^2)}\, dx$.