Part IB: Physics B (Electromagnetism)
Examples Sheet 1: Electrostatic force and potential, electric dipoles, and Gauss' law

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Questions marked with a $(*)$ are difficult and should not be attempted at the expense of the other questions.

Maxwell's equations & the Lorentz force law (per unit volume)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

Basic concepts: charge, superposition and force

1. Carefully state *Coulomb's law*, in *vector* form, for the electric field $\mathbf{E}(\mathbf{x})$ at position $\mathbf{x} \in \mathbb{R}^3$ due to a charge q fixed at position $\mathbf{x}_0 \in \mathbb{R}^3$. Hence write down Coulomb's law in (a) Cartesian coordinates; (b) spherical coordinates centred on the fixed charge q at $\mathbf{x}_0$. Draw sketches of the electric field due to a point charge in the cases where $q > 0$ and $q < 0$.

2. (a) Carefully state the *principle of superposition* for electric fields. Where does this principle come from?
   (b) Using the principle of superposition and Coulomb's law, write down an equation for the electric field $\mathbf{E}(\mathbf{x})$ at position $\mathbf{x} \in \mathbb{R}^3$ due to two charges q_1, q_2 at positions $\mathbf{x}_1, \mathbf{x}_2$ respectively.
   (c) In the case $q = q_1 = q_2$, deduce that the electric field is zero *only* at the midpoint, $\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$. Give a sketch of the electric field $\mathbf{E}(\mathbf{x})$ in this case.

3. Using the principle of superposition, and *without* assuming any symmetry of the system, show that the electric field due to an infinite line charge, of charge λ per unit length, is:

$$\mathbf{E}(r, \theta, z) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{e}}_r,$$

   where (r, θ, z) are cylindrical coordinates, with the line charge itself taken as the z-axis. *[Hint: break the line charge into small elements of charge λdz, then integrate to sum the contributions to the electric field at a point (r, θ, z) from all of these small elements of charge.]*

4. Consider a system comprising a point charge q a distance d from an infinite line charge, of charge λ per unit length. Compute:
   (a) the force on the point charge due to the line charge;
   (b) the force on the line charge due to the point charge;
   (c) the force exerted on the line charge by itself;
   (d) the force exerted on the point charge by itself.

Electrostatic potential and electrostatic energy

5. (a) Explain what it means for: (i) the electric field $\mathbf{E}(\mathbf{x})$ to be *conservative* (defining an *electrostatic potential* in your answer); (ii) the electric field $\mathbf{E}(\mathbf{x})$ to be *irrotational*. State the relationship between these two conditions.
   (b) Show directly that the electric field due to a point charge is both conservative and irrotational.
   (c) Show directly that the electric field due to an infinite line charge, of charge λ per unit length, is both conservative and irrotational.

6. Sketch the electric field lines and equipotential surfaces that arise from the potential $\phi(r, \theta, z) = Ar^2 \cos(2\theta)$ in cylindrical coordinates (r, θ, z), where A is a constant.

7. (a) Define the *potential difference* between the points $\mathbf{x}_1, \mathbf{x}_2$ and give formulae for it in terms of both: (i) the electric potential; (ii) the electric field. Prove that your formulae are equivalent.

   (b) State the electrostatic energy of a charge q at position $\mathbf{x}_0$ in an electric field with potential $\phi(\mathbf{x})$. Similarly, state the work done by an electric field with potential $\phi(\mathbf{x})$ in moving a charge q from position $\mathbf{x}_1$ to position $\mathbf{x}_2$.

   (c) An electron is accelerated through a potential difference of 150 V. What is its change in speed?

Electric dipoles

8. (a) Using the principle of superposition, write down the electric field $\mathbf{E}(\mathbf{x})$ and (an) electric potential due to a *finite electric dipole*, comprising a charge $-q$ at $\mathbf{0}$ and a charge $+q$ at $\mathbf{a}$. Sketch the electric field and equipotential surfaces due to the dipole.

   (b) Show that on scales where the separation distance is negligible, $|\mathbf{x}| \gg |\mathbf{a}|$, the electric field and (an) electric potential approach:
   $$\mathbf{E}(\mathbf{x}) = \frac{3(\mathbf{p}\cdot\mathbf{x})\mathbf{x} - |\mathbf{x}|^2\mathbf{p}}{4\pi\epsilon_0|\mathbf{x}|^5}, \qquad \phi(\mathbf{x}) = \frac{\mathbf{p}\cdot\mathbf{x}}{4\pi\epsilon_0|\mathbf{x}|^3},$$
   where $\mathbf{p} = q\mathbf{a}$. [*Hint: consider the binomial expansion of* $|\mathbf{x}+\mathbf{a}|^{-1} = \left(\mathbf{x}^2 + \mathbf{a}^2 - 2\mathbf{a}\cdot\mathbf{x}\right)^{-1/2}$.] Express these formulae in: (i) Cartesian coordinates; (ii) spherical coordinates, with the z-axis oriented along $\hat{\mathbf{p}}$ in both cases.

9. (a) Show that the force on an infinitesimal electric dipole $\mathbf{p}$ placed in an electric field $\mathbf{E}(\mathbf{x})$ at the point $\mathbf{x}_0$ is given by $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})(\mathbf{x}_0) = ((\mathbf{p}\cdot\nabla)\mathbf{E})(\mathbf{x}_0)$, justifying the equivalence of the two formulae.

   (b) Show that the electrostatic energy of an infinitesimal electric dipole $\mathbf{p}$ placed in an electric field $\mathbf{E}(\mathbf{x})$ at the point $\mathbf{x}_0$ is given by $U = -\mathbf{p}\cdot\mathbf{E}(\mathbf{x}_0)$.

   (c) Show that the torque about the origin on an infinitesimal electric dipole $\mathbf{p}$ placed in an electric field $\mathbf{E}(\mathbf{x})$ at the point $\mathbf{x}_0$ is given by $\mathbf{G} = \mathbf{p}\times\mathbf{E}(\mathbf{x}_0) + \mathbf{x}_0\times\mathbf{F}$, where $\mathbf{F}$ is the force obtained in (b). Why does the standard formula $\mathbf{G} = \mathbf{x}_0\times\mathbf{F}$ fail here?

   (d) Describe the motion of an electric dipole $\mathbf{p}$, of moment of inertia I about an axis going through its centre of mass and perpendicular to the plane of the dipole, when it is immersed in a constant electric field $\mathbf{E}_0$ parallel to the plane of the dipole, supposing that the dipole and electric field are initially inclined at a small angle $\theta_0 \ll 1$.

10. Two coplanar identical infinitesimal electric dipoles of dipole moment p are supported on pivots a large distance d apart. Each dipole can rotate only in the plane. Their angles of twist, θ_1 and θ_2, are measured clockwise from the line joining their centres. Show that the electrostatic energy of one dipole in the field of the other is:
$$-\frac{p^2}{8\pi\epsilon_0 d^3}\left(3\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2)\right).$$

Charge conservation

11. (a) State and prove the *continuity equation* describing the conservation of charge. You should explain the meaning of the terms *charge density* ρ and *current density* $\mathbf{J}$ in your answer.

    (b) Consider a straight cylindrical wire, oriented along the z-axis. Suppose that n electrons pass a cross-section of the wire per unit time, with velocity tangential to the wire. Explain why $\mathbf{J} = -ne\hat{\mathbf{e}}_z$ inside the wire. What does the continuity equation imply about the charge density along the wire? Does this make physical sense?

12. Suppose that the current density takes the form $\mathbf{J}(\mathbf{x}, t) = C\mathbf{x}e^{-at|\mathbf{x}|^2}$, where C and a are fixed constants. Show that charge is conserved if the charge density takes the form $\rho(\mathbf{x}, t) = (f(\mathbf{x}) + tg(\mathbf{x}))\,e^{-at|\mathbf{x}|^2}$, where f, g are functions you should determine.

13. In a fluid, charge undergoes *diffusion*. This is described empirically by *Fick's law* $\mathbf{J} = -D\nabla\rho$, where D is a constant called the *diffusion coefficient*.

    (a) Give a physical explanation for Fick's law.

    (b) Show that in a fluid, the charge density ρ satisfies the diffusion equation. Verify that a spreading Gaussian of the form:
    $$\rho(\mathbf{x},t) = \frac{\rho_0 a^3}{(4Dt)^{3/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4Dt}\right)$$
    solves this equation, and give a physical interpretation of this result.

Gauss' law

14. State Gauss' law in both *integral form* and *differential form*, defining all the quantities that arise in your statements. Prove that the integral form and differential form are equivalent to one another. Does Gauss' law hold for time-varying electric fields?

15. (a) For an infinite line charge, of charge λ per unit length, argue by symmetry that the electric field takes the form $\mathbf{E}(r,\theta,z) = E(r)\hat{\mathbf{e}}_r$, where (r,θ,z) are cylindrical coordinates, with the line charge itself taken as the z-axis.

    (b) Hence, use Gauss' law to compute the electric field $\mathbf{E}(r,\theta,z)$ everywhere due to the infinite line charge, and verify that your answer agrees with Question 3. Which derivation is more straightforward?

16. Compute and sketch the electric field $\mathbf{E}(\mathbf{x})$ and electric potential $\phi(\mathbf{x})$ everywhere due to an infinite plane carrying a surface charge σ per unit area. Hence compute and sketch the electric field $\mathbf{E}(\mathbf{x})$ and electric potential $\phi(\mathbf{x})$ everywhere due to:

    (a) two parallel infinite planes of separation distance d, carrying equal and opposite surface charges $\sigma, -\sigma$ per unit area;

    (b) two infinite planes intersecting at an angle $\pi/4$, carrying equal and opposite surface charges $\sigma, -\sigma$ per unit area.

17. Compute and sketch the electric field $\mathbf{E}(\mathbf{x})$ and electric potential $\phi(\mathbf{x})$ everywhere due to a a spherical shell of negligible thickness and radius a, carrying a total charge Q.

18. (a) Compute and sketch the electric field $\mathbf{E}(\mathbf{x})$ and electric potential $\phi(\mathbf{x})$ everywhere due to a spherical ball of radius a carrying a uniform total charge Q.

    (b) Hence, compute and sketch the electric field $\mathbf{E}(\mathbf{x})$ and electric potential $\phi(\mathbf{x})$ everywhere due to a thick spherical shell, with inner radius a_1 and outer radius a_2, carrying a uniform total charge Q. [*Hint: superposition.*]

    (c) Show also that as the thickness tends to zero, but the total charge on the shell remains constant, we recover the correct electric field and electric potential due to a spherical shell of negligible thickness, as computed in Question 17.

19. ($*$) Using the results of the previous question, compute the force on one of the hemispheres of a spherical ball due to the entire sphere, which itself is of radius a and uniform total charge Q.

20. Compute and sketch the electric field $\mathbf{E}(\mathbf{x})$ everywhere due to the non-uniform, spherically symmetric charge distribution:
    $$\rho(r,\theta,\phi) = \begin{cases} \rho_0\left(1 - \dfrac{r}{R}\right), & \text{for } r \leq R, \\ 0, & \text{for } r > R, \end{cases}$$
    where ρ_0, R are constants, and (r,θ,ϕ) are spherical coordinates.

21. Compute and sketch the electric field $\mathbf{E}(\mathbf{x})$ everywhere due to the non-uniform charge density $\rho(x,y,z) = \rho_0 e^{-k|z|}$, where ρ_0, k are constants, and (x,y,z) are Cartesian coordinates.