Part IB: Physics B (Electromagnetism)
Examples Sheet 2: Poisson's equation and the method of images

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Questions marked with a ($*$) are difficult and should not be attempted at the expense of the other questions.

Maxwell's equations & the Lorentz force law (per unit volume)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

Conductors and boundary conditions

1. Explain why on the interior of an *electrostatic* conductor we must have that the electric field is zero, $\mathbf{E} = \mathbf{0}$. Hence show:
   (a) within a conductor, any electric potential must be constant;
   (b) within a conductor, the charge density must be zero;
   (c) any charge on a conductor must reside on its surface.

2. Consider a thin surface S bounding a volume V.
   (a) If the value of the electric potential (relative to the ground, for example) is specified on the surface S, what is the name of the type of 'boundary conditions' that apply to the volume V?
   (b) Let σ be the (possibly non-uniform) surface charge density on S. Use Maxwell's equations for electrostatics to show that the electric field experiences a discontinuity:
   $$\mathbf{E}_+ - \mathbf{E}_- = \frac{\sigma \hat{\mathbf{n}}}{\epsilon_0}$$
   across S, where $\mathbf{E}_+$ is the electric field just outside of S, $\mathbf{E}_-$ is the electric field just inside of S, and $\hat{\mathbf{n}}$ is an outward-pointing normal. Show that the electric potential is continuous across S, but that the gradient of the electric potential is similarly discontinuous:
   $$\nabla \phi_+ - \nabla \phi_- = -\frac{\sigma \hat{\mathbf{n}}}{\epsilon_0}.$$
   If the value of σ is specified on the surface S, what is the name of the type of 'boundary conditions' that apply to the volume V?

Poisson's equation

3. State Maxwell's equations for electrostatics. Using these equations show that an electric potential ϕ for an electrostatic system with charge density $\rho(\mathbf{x})$ must satisfy *Poisson's equation* $\nabla^2 \phi = -\rho(\mathbf{x})/\epsilon_0$.

4. (a) Show that in *free space* (i.e. requiring $\phi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$), the solution of Poisson's equation is given by:
   $$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int\limits_{\mathbb{R}^3} \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, dV.$$
   [*You may quote a relevant Green's function for Poisson's equation, if its meaning is clearly explained.*] Deduce a general formula for the electric field in free space, given a charge density ρ. Give an interpretation of this formula.
   (b) Using the Green's function solution to Poisson's equation in free space, compute the electric field at a height z above the axis of a disk of radius a, carrying surface charge σ per unit area. Show that the electric field experiences the expected discontinuity across the disk.

5. Prove that the solution to Poisson's equation in a bounded volume V with boundary S is unique, provided that either: (i) Dirichlet boundary conditions are specified on the boundary; (ii) Neumann boundary conditions are specified on the boundary. Why can't the Green's function solution be used in this case?

6. By guessing a solution to Poisson's equation, prove that inside an empty conducting surface S, the electric field must always vanish. Comment on this result in relation to *Faraday cages*.

7. (a) Consider two parallel infinite charged plates separated by a distance d kept at a constant potential difference V. Compute the electric field everywhere by guessing a solution to Poisson's equation, with appropriate boundary conditions at the plates. Sketch the electric field lines and the equipotential surfaces. Compute also the surface charge per unit area on each of the plates.

   (b) Consider two parallel infinite charged plates separated by a distance d carrying uniform charges $+\sigma$ and $-\sigma$ per unit area, respectively. Compute the resulting electric field everywhere by guessing a solution to Poisson's equation, with appropriate boundary conditions at the plates. Sketch the electric field lines and the equipotential surfaces. Compute also the potential difference between the plates.

   (c) Comment on the relationship between (a) and (b). Is this a special case, or general behaviour?

8. Consider a neutral spherical conductor, centred at the origin and of radius a, which is immersed in a constant electric field $\mathbf{E} = E\hat{\mathbf{e}}_z$.

   (a) What are the boundary conditions for this problem? Are they Dirichlet, Neumann, or both?

   (b) By guessing a solution to Poisson's equation, show that the electric field everywhere is given by:

$$\mathbf{E}(r,\theta,\phi) = \begin{cases} \mathbf{0}, & \text{if } r < a, \\ \left(\dfrac{2a^3}{r^3} + 1\right) E\cos(\theta)\hat{\mathbf{e}}_r + \left(\dfrac{a^3}{r^3} - 1\right) E\sin(\theta)\hat{\mathbf{e}}_\theta, & \text{if } r > a, \end{cases}$$

   in spherical polar coordinates. [*Hint: consider a potential which is the sum of a dipole potential and uniform potential.*] Sketch the electric field lines and equipotential surfaces.

   (c) Compute the surface charge density on the spherical conductor.

9. Repeat Question 8 for an infinite cylindrical conductor of radius a with axis along the x-axis, immersed in a constant electric field $\mathbf{E} = E\hat{\mathbf{e}}_z$.

10. Two point charges q and $-q$ are a distance $2d$ apart. Show that, if a conducting sphere of small radius a ($a \ll 2d$) is placed midway between the two charges, the force on each is increased by a factor of approximately $1 + 16a^3/d^3$.

11. Consider two parallel infinite conducting plates separated by a gap d, where one plate has a small hemispherical protrusion of radius $a \ll d$, which is directed into the gap. Suppose that a potential difference V is maintained across the plates.

    (a) What are the boundary conditions for this problem? Are they Dirichlet, Neumann, or both?

    (b) By guessing a solution to Poisson's equation, show that the electric field in the region between the plates is given *approximately* by:

$$\mathbf{E}(r,\theta,\phi) = -\frac{V}{d}\left(1 + \frac{2a^3}{r^3}\right)\cos(\theta)\hat{\mathbf{e}}_r + \frac{V}{d}\left(1 - \frac{a^3}{r^3}\right)\sin(\theta)\hat{\mathbf{e}}_\theta,$$

    in spherical coordinates, where the centre of coordinates is taken to be the centre of the hemispherical protrusion, and the z axis is oriented along the axis of symmetry of the problem, towards the gap. [*Hint: consider a potential which is the sum of a dipole potential and uniform potential.*] Sketch the electric field lines and equipotential surfaces.

    (c) Compute the surface charge density on the plate with the hemispherical protrusion. [*Hint: you will need to treat the charge on the hemispherical section and the charge on the flat section separately.*]

The method of images

12. **Essay question.** You might be surprised and disappointed to hear that IB Physics has *essay questions* for electromagnetism - so, time to get some practice! Write a short essay (no more than two pages) on the theoretical basis of the method of images in electrostatics. Give two basic examples of applications of the method of images (which, preferably, are *not* ones from the examples sheet), demonstrating that you understand the technique.

13. The electrical system of a typical thundercloud can be represented by a vertical (finite) dipole consisting of a charge of $+40$ C at a height of 10 km and a charge of -40 C vertically below it at a height of 6 km. The ground may be treated as a conductor.

    (a) What is the electric field at the ground immediately below the thundercloud?

    (b) At what distance from the axis of the thundercloud is the field at the ground zero?

14. Two conducting plates are placed along the planes $x = 0$ and $y = 0$. A charge is placed at the point (x, y) with $x, y > 0$. By placing appropriate image charges, determine:

    (a) the electric field in the region $x, y > 0$;

    (b) the induced surface charge per unit area on each of the plates.

15. Two infinite conducting plane sheets meet at an angle of $\pi/3$. A charge q is constrained to move on the plane bisecting this angle. Show that the force on the particle when it is a distance x from the line on which the sheets meet is given by:

$$\frac{q^2}{4\pi\epsilon_0 x^2}\left(\frac{5}{4} - \frac{1}{\sqrt{3}}\right),$$

    directed towards the plates.

16. A charge q is placed inside a spherical conductor of radius a at a distance $0 < d < a$ from the centre. By placing an appropriate image charge outside of the conductor, compute the force on the charge. [*Hint: try varying both the strength and position of the image charge.*] Compute also the induced (non-uniform) surface charge density on the sphere.

17. An infinitely long cylinder of radius a, carrying charge λ per unit length, is placed with its axis parallel and at a distance d to an infinite conducting plane, which is grounded at zero potential. By considering two appropriate image line charges, show that the potential of the cylinder is:

$$\frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{d + \sqrt{d^2 - a^2}}{a}\right).$$

    What is the force exerted on the cylinder by the conducting plane?

18. An infinitesimal dipole $\mathbf{p} = p\hat{\mathbf{e}}_z$ is placed at a height h above a grounded conducting plane at $z = 0$. Determine the electric field due to the dipole, and the induced surface charge on the conducting plate.

19. ($*$) Find the electric field due to an arbitrary charge density $\rho(\mathbf{x})$ occupying the region $z \geq 0$, when an infinite conducting plate is placed along $z = 0$.