Part IB: Physics B (Electromagnetism)
Examples Sheet 3: Capacitors, electrostatic energy, virtual work, and dielectrics

Please send all comments and corrections to jmm232@cam.ac.uk.

Questions marked with (∗) are difficult and should not be attempted at the expense of the other questions.

Maxwell's equations & the Lorentz force law (per unit volume)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

Capacitors

1. What is a *capacitor*? Define the *mutual capacitance* of a capacitor, and show that is a function only of the geometry of the capacitor. Compute the mutual capacitance of:
   (a) two parallel infinite plates of separation distance d, per unit area;
   (b) two cylindrical shells with a common axis, of radii a, b, per unit length;
   (c) two parallel cylinders of radius a, with axes separated by a distance $d > 2a$, per unit length;
   (d) two spherical shells with a common centre, of radii a, b.

2. Consider a parallel plate capacitor comprising two *finite* square plates, of area A, with small separation distance d.
   (a) Sketch the electric field lines associated with the capacitor, both internally and externally, giving justification for your sketch.
   (b) Show that the electric field strength at a midpoint of one of the sides of the capacitor, and equidistant from both capacitors, is approximately half the electric field strength at the centre of the capacitor. What is the relation of the electric field strength at the corners to the centre? [*Hint: consider joining two such capacitors.*]

Electrostatic energy and virtual work

3. (a) Suppose that charges $q_1, ..., q_n$ are placed at positions $\mathbf{x}_1, ..., \mathbf{x}_n \in \mathbb{R}^3$. Explain (with particular reference to the factor of $1/2$) why the electrostatic energy of the resulting system is, when self-energy is neglected:
   $$U = \frac{1}{2} \sum_{i=1}^{n} q_i \phi_i(\mathbf{x}_i),$$
   where $\phi_i(\mathbf{x})$ is the potential due to all the charges *excluding* q_i.
   (b) Three positive charges q_1, q_2, q_3 are constrained to move on the x-axis, with positions x_1, x_2, x_3 respectively. Write down their electrostatic energy. If q_1, q_3 are fixed at x_1, x_3 respectively, determine the location x_2 which minimises the electrostatic energy of the system.

4. By taking the continuum limit of the previous question, show that the electrostatic energy of a continuous charge distribution $\rho(\mathbf{x})$ producing an electric field with electric potential $\phi(\mathbf{x})$ is given by:
   $$U = \frac{1}{2} \int_{\mathbb{R}^3} \rho(\mathbf{x}) \phi(\mathbf{x}) \, dV.$$
   (∗) Does this formula include or exclude self-energy? Is it consistent with the formula in the previous question?

5. Prove that the electrostatic energy stored on the plates of a capacitor of capacitance C, held at a potential difference V and carrying a total charge Q on one plate, is given by $\frac{1}{2}CV^2 = \frac{1}{2}QV = Q^2/2C$ using the formula for electrostatic energy we derived in the previous question.

6. Two isolated spherical conductors of radii a and b are charged to potentials V_a and V_b respectively. They are then connected by a fine resistive wire. How much heat will be generated in the wire in total, if its resistance is sufficient to overdamp the system?

7. (a) Using Gauss' law, show that the electrostatic energy of a charge distribution producing electric field $\mathbf{E}(\mathbf{x})$ may be written as:
$$U = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} |\mathbf{E}|^2 \, dV.$$
Hence explain the phrase '*energy density of the electric field*', and explain why this concept is useful.
   (b) Compute the electrostatic energy of a spherical ball of radius a carrying a uniform total charge Q by: (i) using the charge density and electric potential; (ii) using the electric field. Verify that your answers agree.
   (c) Hence, compute a 'classical radius' for the electron, under the dubious assumption that its rest mass results entirely from electrostatic energy.

Virtual work

8. State and prove the *principle of virtual work*. Using this principle, prove the *law of the lever*: the ratio of the resulting force on the load to the applied force on the lever is equal to the ratio of the distance of the user from the fulcrum to the distance of the load from the fulcrum.

9. Compute the force between the two plates of a parallel plate capacitor by considering separating the plates, (a) with the charges on the plates kept constant; (b) with the potential difference across the plates kept constant. Verify that your results are correct by computing the force between the plates directly using the Lorentz force law.

Dielectric media: basics and Gauss' law

10. (a) Show that the charge density of an infinitesimal dipole $\mathbf{p}$ placed at $\mathbf{x}_0$ is given by:
$$\rho(\mathbf{x}) = -\mathbf{p} \cdot \nabla \left(\delta^3(\mathbf{x} - \mathbf{x}_0)\right) = -\nabla \cdot \left(\mathbf{p}\delta^3(\mathbf{x} - \mathbf{x}_0)\right),$$
where you should justify the equivalence of the two given forms. Interpret this result. [*Hint: the charge density of a point charge q at position $\mathbf{x}_0$ is $q\delta^3(\mathbf{x} - \mathbf{x}_0)$ - why? You may assume all manipulations of the delta function.*]
   (b) Explain the difference between *free* and *bound* charge densities, ρ_{free}, ρ_{bound}, and explain the meaning of the *polarisation density* $\mathbf{P}(\mathbf{x})$. Using part (a), explain why $\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$, and hence show that the bound charge density that accumulates on the surface of a dielectric is given by $\mathbf{P} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit normal to the dielectric.
   (c) Using part (b), obtain Gauss' law in the form $\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\text{free}}$. Hence, define the electric displacement $\mathbf{D}$, and explain its mathematical significance. Does it have a physical interpretation? Is $\mathbf{D}$ a conservative vector field?
   (d) How are $\mathbf{E}$, $\mathbf{P}$ and $\mathbf{D}$ related in a *linear, isotropic, homogeneous* dielectric medium? Give an example of a material which is a dielectric, but *not* a *linear, isotropic, homogeneous* dielectric.

11. A charge q is placed at the centre of a linear dielectric sphere, of radius a and relative permittivity ϵ. Determine the electric displacement, electric field, and polarisation density everywhere, and sketch them. Comment on the location of the bound charge density.

12. An infinite, thin wire carrying uniform charge λ per unit length is surrounded by a cylindrical dielectric medium of radius a and relative permittivity ϵ. Determine the electric displacement, electric field, and polarisation density everywhere, and sketch them. Comment on the location of the bound charge density.

Dielectric media: boundary value problems

13. Show that at a boundary carrying free surface charge, the electric displacement experiences a normal discontinuity $(\mathbf{D}_+ - \mathbf{D}_-) \cdot \hat{\mathbf{n}} = \sigma_f$, where σ_f is free surface charge density. Deduce that the electric displacement is normally continuous, in the absence of free charge. Is the electric displacement tangentially continuous at a boundary?

14. Determine the electric fields everywhere when we immerse in a uniform electric field $\mathbf{E}_0$:

    (a) an infinite dielectric slab of thickness h and relative permittivity ϵ, perpendicular to $\mathbf{E}_0$;
    (b) an infinite dielectric slab of thickness h and relative permittivity ϵ, at an angle $\pi/4$ to $\mathbf{E}_0$;
    (c) a dielectric sphere of radius a and relative permittivity ϵ;
    (d) a dielectric cylinder of radius a and relative permittivity ϵ.

    In each case, sketch the resulting electric fields $\mathbf{E}$, electric displacements $\mathbf{D}$, and polarisation densities $\mathbf{P}$.

15. Consider a dielectric *hemisphere*, of radius a and relative permittivity ϵ, placed with its flat side on the surface of an infinite conducting plane. Suppose that the entire system is immersed in a constant electric field $\mathbf{E}_0$ which is perpendicular to the plane. Using the result from Question 16(c), find (a) the additional charge that appears under the hemisphere; (b) the additional charge that appears on the rest of the plane.

Dielectric media: capacitor problems

16. An infinite coaxial cable is constructed using two cylindrical conductors, with radii 1 and 7 mm. Two cylindrical dielectrics fill the region between the conductors. The inner dieletric occupies one sixth of the total area between the conductors. The relative permittivities of the dielectrics are $\epsilon_{\text{in}} = 2$, $\epsilon_{\text{out}} = 3$. Show that the capacitance per unit length of the cable is 110 pF m^{-1}.

17. Consider a parallel plate capacitor, where the plates have large area A and are separated by a small distance d. Suppose that a cuboid of height d, occupying a fraction r of the capacitor, is filled with a linear dielectric of relative permittivity ϵ; suppose also that the remainder of the capacitor is filled with vacuum.

    (a) Show carefully that the capacitance of the system is $(1 - r + r\epsilon)\,\epsilon_0 A/d$. Comment on the relationship of this result to adding capacitances in parallel.
    (b) Sketch the electric field due to the free charge on the plates, and hence explain which direction the electrostatic force will act on the dielectric. Determine the magnitude of the electrostatic pressure exerted on the dielectric using (i) a virtual work argument; (ii) ($*$) a direct force argument.

18. Suppose that the region occupied by the dielectric in the previous question still constitutes a fraction r of the gap between the plates, but is now not necessarily a cuboid of height h. Calculate the ratio of the maximum possible capacitance of the system to the minimum possible capacitance of the system, as the occupied region varies.

19. (a) Consider a concentric cylindrical capacitor comprising two conducting cylinders of radii $a < b$ of long length L. The capacitor is immersed in a linear dielectric liquid of density ρ and relative permittivity ϵ up to a height h. Show that the capacitance of the resulting system is:

    $$\frac{2\pi\epsilon_0}{\log(b/a)}(L - h + \epsilon h).$$

    (b) Let V be the potential difference across the capacitor. Using a virtual work argument, or otherwise, show that the equilibrium height of the liquid is given by:

    $$h = \frac{\epsilon_0(\epsilon - 1)V^2}{g\rho(b^2 - a^2)\ln(b/a)}.$$

Dielectric media: energy

20. Prove that the energy stored in an electrostatic field may be rewritten in the form:

    $$U = \frac{1}{2}\int_{\mathbb{R}^3} \mathbf{D}\cdot\mathbf{E}\,dV.$$

    Using this formula, find the energy stored in the capacitors in Questions 16, 17, 18 and 19, in each case comparing to the formula $\frac{1}{2}CV^2$.