Part IB: Physics B (Electromagnetism)
Examples Sheet 4: The Biot-Savart law, magnetic dipoles, and Ampère's law

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Questions marked with ($*$) are difficult and should not be attempted at the expense of the other questions.

Maxwell's equations & the Lorentz force law (per unit volume)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

The Biot-Savart law

1. Carefully state the *Biot-Savart law* for the magnetic field $\mathbf{B}(\mathbf{x})$ at position $\mathbf{x} \in \mathbb{R}^3$ due to a thin, stationary wire described by a closed curve C carrying a static current I. Using the Biot-Savart law, and in each case assuming the wire carries a static current I, compute the magnetic field:
   (a) everywhere, due to a thin infinite wire along the z-axis;
   (b) everywhere due to an infinite cylindrical solenoid having n turns per unit length;
   (c) a distance z along the axis of a circular current loop of radius a;
   (d) a distance z along the axis from the end of a semi-infinite solenoid having n turns per unit length;
   (e) at the centre of a square current loop of side length a.
2. Using the Biot-Savart law, calculate the approximate magnetic field due to a small rectangular current loop of side lengths Δx, Δy: (a) a distance z from the centre of the loop, in a direction perpendicular to the plane of the loop; (b) a distance x from the centre of the loop, in a direction parallel to one of the sides of the loop.
3. How should the Biot-Savart law modified to give the magnetic field $\mathbf{B}(\mathbf{x})$ due to a current *density* $\mathbf{J}(\mathbf{x})$ occupying a region V? Using this version of the law, compute the magnetic field everywhere due to a thick cylindrical wire, of radius a, carrying a current I uniformly distributed across its cross-section. Compare your result with Question 1(a).

Force and torques on currents

4. (a) Show that in a uniform magnetic field, the force on a current loop C is always zero.
   (b) Show that in the non-uniform magnetic field $\mathbf{B}(x, y, z) = B_0(x, y, 0)$ the force on a circular current loop at $x^2 + y^2 = a^2$, $z = 0$ carrying current I has magnitude $2\pi I a^2 B_0$. In what direction does the force act?
   (c) How is it possible that a current loop can experience a force, if magnetic fields do no work?
5. A cylindrical column of mercury of radius a carries a current I, uniformly distributed over its cross-section. By considering the force per unit volume on the current, derive a formula for the pressure p as a function of radius r in the column, ignoring the weight of the mercury.
6. (a) Show that the total force exerted by a wire C_1 carrying current I_1 on a wire C_2 carrying current I_2 is given by the nested line integral:
   $$\mathbf{F}_{12} = \frac{I_1 I_2 \mu_0}{4\pi} \oint_{C_2} d\mathbf{x}_2 \times \left(\oint_{C_1} d\mathbf{x}_1 \times \frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \right).$$
   When is this force attractive, and when is it repulsive? [*Bonus: is the formula (4.6) in the lecture notes correct?*]
   (b) Find the magnitude and direction of the force per unit length between two infinitely long, parallel wires a distance d apart, carrying currents I_1, I_2 respectively.
   (c) ($*$) Prove directly that the force between currents respects Newton's third law.

7. Explain why the torque about the origin on a current loop C carrying current I in a magnetic field **B** may be expressed as the line integral:

$$\mathbf{G} = I \oint_C \mathbf{x} \times (d\mathbf{x} \times \mathbf{B})\,.$$

   Hence calculate the torque about the centre of a circular current loop of radius a carrying current I immersed in a constant magnetic field **B**, when the plane of the loop makes an angle θ with the direction of the magnetic field.

Gauss' law for magnetism

8. State *Gauss' law for magnetism* in both integral and differential form, and prove their equivalence. What is the physical relevance of this law? Verify the law holds true for the magnetic fields in Questions 1(a) and 1(b).

9. Using Gauss' law for magnetism, show that at a distance z along the axis of a circular current loop of radius a, the radial component of the magnetic field at a small distance r from the axis is approximately:

$$\frac{3\mu_0 I a^2 z r}{4(z^2 + a^2)^{5/2}}.$$

The magnetic scalar potential

10. Explain why, away from currents, we may write $\mathbf{B}(\mathbf{x}) = -\mu_0 \nabla \phi_m$, where ϕ_m is a scalar function called the *magnetic scalar potential*. Does this potential have any physical significance?

11. Define the *solid angle* $\Omega(\mathbf{x})$ of a surface S as seen from a point **x**, and explain its geometrical significance. Show, *directly from the Biot-Savart law*, that the magnetic scalar potential due to a loop carrying current I is:

$$\phi_m(\mathbf{x}) = \frac{I\Omega(\mathbf{x})}{4\pi},$$

   where $\Omega(\mathbf{x})$ is the solid angle of the loop viewed from the point **x**. [*Hint: you may find it useful to use the vector identity* $(\mathbf{v} \times \nabla) \times \mathbf{w} = \mathbf{v} \times (\nabla \times \mathbf{w}) + (\mathbf{v} \cdot \nabla)\mathbf{w} - \mathbf{v}(\nabla \cdot \mathbf{w})$. *You may even like to prove it!*]

Magnetic dipoles

12. (a) Give a rough sketch of the magnetic field lines due to an (arbitrary shape) current loop carrying current I, explaining the main features.

    (b) Show that on scales where the size of a current loop centred at the origin is negligible, $|\mathbf{x}| \gg a$ (with a some characteristic size of the loop), the magnetic field and (a) magnetic scalar potential approach:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0(3(\mathbf{m} \cdot \mathbf{x})\mathbf{x} - |\mathbf{x}|^2 \mathbf{m})}{4\pi |\mathbf{x}|^5}, \qquad \phi_m(\mathbf{x}) = \frac{\mathbf{m} \cdot \mathbf{x}}{4\pi |\mathbf{x}|^3},$$

    where $\mathbf{m} = I\mathbf{S}$, with **S** the vector area of the current loop. What do these formulae remind you of?

    (c) Compute the magnetic dipole moment of the small loop in Question 2. Are the expressions you found for the magnetic field consistent with the formulae in part (b)?

13. (a) Show that the force on an infinitesimal magnetic dipole **m** placed in a magnetic field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}_0$ is given by $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})(\mathbf{x}_0) = ((\mathbf{m} \cdot \nabla)\mathbf{B})(\mathbf{x}_0)$, justifying the equivalence of the two formulae.

    (b) Show that the magnetostatic energy of an infinitesimal magnetic dipole **m** placed in a magnetic field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}_0$ is given by $U = -\mathbf{m} \cdot \mathbf{B}(\mathbf{x}_0)$. How can this be, if magnetic fields do no work?

    (c) Show that the torque about the origin on an infinitesimal magnetic dipole **m** placed in a magnetic field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}_0$ is given by $\mathbf{G} = \mathbf{m} \times \mathbf{B} + \mathbf{x}_0 \times \mathbf{F}$, where **F** is the force obtained in (b).

    (d) A magnetic dipole **m** is placed at a distance z along the axis from the end of a semi-infinite cylindrical solenoid of radius a, with n turns per unit length. Calculate the force on the magnetic dipole when a current I flows in the solenoid.

Ampère's law

14. State Ampère's law in both *integral form* and *differential form*, defining all the quantities that arise in your statements. Prove that the integral form and differential form are equivalent to one another.

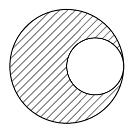
15. (a) For an infinite wire carrying current I, argue by symmetry that the magnetic field must take the form $\mathbf{B}(r, \theta, z) = B(r)\hat{\mathbf{e}}_\theta$, where (r, θ, z) are cylindrical coordinates, with the wire itself taken as the z-axis. Why does the functional form differ from the functional form of the electric field ? [*Hint: What happens to the magnetic field when it is reflected in a mirror?*]

    (b) Hence, use Ampère's law to compute the magnetic field $\mathbf{B}(r, \theta, z)$ everywhere due to the infinite wire, and verify that your answer agrees with Question 1(a). Which derivation is more straightforward?

16. Use Ampère's law to derive the magnetic field everywhere due to an infinite cylindrical solenoid, of radius a, having n turns per unit length. Verify that your answer agrees with Question 1(b).

17. (a) Use Ampère's law to derive the magnetic field everywhere due to an infinite thick cylindrical wire of radius a, carrying a uniform static current I distributed uniformly across its cross-section. Verify that your answer agrees with Question 3.

    (b) Consider a thick cylindrical wire whose cross-section takes the form shown in the figure below.



The inner cylinder has half the diameter of the outer cylinder and is empty, whilst the region between the cylinders is filled with a uniform current density along the direction of the wire. Using part (a), and the principle of superposition, calculate the magnetic field everywhere.

The magnetic vector potential

18. Define the *magnetic vector potential*, and explain the freedom we have when choosing it. Does it have any physical significance? Show that the following potentials all give rise to the same magnetic field:

$$\mathbf{A}_1 = xB\hat{\mathbf{e}}_y, \qquad \mathbf{A}_2 = \frac{1}{2}B\left(x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x\right), \qquad \mathbf{A}_3 = \frac{1}{2}rB\hat{\mathbf{e}}_\theta, \qquad \mathbf{A}_4 = \frac{1}{2}r\sin(\theta)B\hat{\mathbf{e}}_\phi,$$

where the first two cases use Cartesian coordinates, the third case uses cylindrical polar coordinates, and the fourth uses spherical polar coordinates.

19. $(*)$ Show that Maxwell's equations for magnetostatics are equivalent to the Biot-Savart law, in the current density form discussed in Question 3. [*Hint: You might like to use a magnetic vector potential satisfying* $\nabla \cdot \mathbf{A} = 0$ *- why can we make this choice?*]

Ohm's law

20. (a) State *Ohm's law* relating the current density, conductivity and electric field, for a static conductor. Is Ohm's law an empirical law, or a fundamental law? Does the law apply for a *moving* conductor?

    (b) Show that Ohm's law can be rewritten in the form $V = IR$, where V is the potential difference across a section of wire, I is the current across the wire, and R is the resistance in the wire (you should define R in terms of the properties of the wire).

21. Show that an arbitrary charge distribution in a dielectric of conductivity σ and relative permittivity ϵ decays with time constant $\tau = \epsilon\epsilon_0/\sigma$.