Part II: Asymptotic Methods - Revision

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1 Definitions and basic properties

1.1 Definitions

Definition: We say $f(x) = O(g(x))$ as $x \to x_0$ if there exist $M > 0$, $\delta > 0$ such that $|x - x_0| \leq \delta$ implies $|f(x)| \leq M|g(x)|$.

Definition: We say $f(x) = o(g(x))$ as $x \to x_0$ if $|f(x)/g(x)| \to 0$ as $x \to x_0$.

Definition: A sequence of functions $\{\phi_k(x)\}_{k=1}^{\infty}$ is called an *asymptotic sequence* as $x \to x_0$ if $\phi_{j+1} = o(\phi_j(x))$ for all j. We say that a function f has an *asymptotic expansion* with respect to the asymptotic sequence $\{\phi_k(x)\}_{k=1}^{\infty}$, written

$$f(x) \sim \sum_{j=0}^{\infty} a_j \phi_j(x) \quad \text{as} \quad x \to x_0,$$

if for all N,

$$\left| f(x) - \sum_{j=0}^{N} a_j \phi_j(x) \right| = o(\phi_N(x)).$$

1.2 Operations on asymptotic expansions

Theorem: The following operations on asymptotic expansions are possible:

(a) If $f(x) \sim \sum a_j \phi_j(x)$ and $g(x) \sim \sum b_j \phi_j(x)$ as $x \to x_0$, then $f(x) + g(x) \sim \sum (a_j + b_j)\phi_j(x)$ as $x \to x_0$.

(b) If $f(x) \sim \sum a_j \phi_j(x)$ and $g(x) \sim \sum b_j \phi_j(x)$ as $x \to x_0$, then

$$f(x)g(x) \sim \sum_{j=0}^{\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k \right) \phi_j(x) \quad \text{as} \quad x \to x_0.$$

(c) Let $f(x) \sim \sum a_j x^j$ as $x \to x_0$. Define

$$F(x) = \int_0^x f(t)dt.$$

Then $F(x) \sim a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3$... as $x \to x_0$, i.e. we can integrate asymptotic expansions term by term.

Proof: (a) and (b) are trivial. For (c), we note:

$$\left| F(x) - \sum_{j=0}^{\infty} \frac{a_j x^{j+1}}{j+1} \right| = \left| \int_0^x f(t)dt - \sum_{j=0}^{N} \int_0^x a_j t^j \, dt \right| \leq \int_0^x \left| f(t) - \sum_{j=0}^{N} a_j t^j \, dt \right|$$

Now let $\epsilon > 0$. Then if $|x|$ is sufficiently small, we have

$$\left| f(t) - \sum_{j=0}^{N} a_j t^j \right| \leq \epsilon |t|^N,$$

and so

$$\int_0^x \left| f(t) - \sum_{j=0}^{N} a_j t^j \, dt \right| \leq \epsilon \int_0^x t^N \, dt \leq \frac{\epsilon |x|^{N+1}}{N+1}. \quad \square$$

1.3 Uniqueness of asymptotic expansions

Theorem: If a function admits an asymptotic expansion using the asymptotic sequence x^j as $x \to 0$, the coefficients of the expansion are unique.

Proof (constructive): By definition, we have

$$f(x) = \sum_{n=0}^{N} a_n x^n + o(x^N)$$

as $x \to 0$, for all N. For $N = 1$, we have $f(x) = a_0 + o(1)$, hence $f(x) \to a_0$ as $x \to 0$, so a_0 determined uniquely. When $N = 1$, $f(x) = a_1 x + a_0 + o(x) \Rightarrow f(x)/x - a_0 = a_1 + o(x)/x$. Hence as $x \to x_0$, we have $f(x)/x - a_0 \to a_1$, so a_1 determined uniquely. Proceed inductively. $\square$

1.4 Stokes' phenomenon

Definition: *Stokes' phenomenon* is the change in asymptotic behaviour of a function defined in the complex plane across specific rays called *Stokes' lines*.

Example: Consider $\sinh(1/z)$ near 0. We have

$$\sinh\left(\frac{1}{z}\right) = \frac{1}{2}\left(e^{(\cos(\theta) - i\sin(\theta))/r} - e^{(-\cos(\theta) + i\sin(\theta))/r}\right).$$

Hence $\sinh(1/z) \sim \frac{1}{2} e^{1/z}$ as $z \to 0$ for $\arg(z) \in (-\pi/2, \pi/2)$ and $\sinh(1/z) \sim -\frac{1}{2} e^{-1/z}$ as $z \to 0$ for $\arg(z) \in (\pi/2, 3\pi/2)$.

2 Integration by parts

Often asymptotic expansions can be obtained simply by integrating by parts.

Example: The asymptotics of

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt,$$

as $x \to \infty$ can be found by repeatedly integrating by parts, giving the result:

$$E_1(x) \sim e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \frac{24}{x^5} ...\right).$$

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3 Watson's Lemma

Theorem (Watson's Lemma): Let $\alpha > -1$, and let g be a continuous function satisfying $|g(t)| \le Ke^{\beta t}$, $t \ge 0$, for some constants β and K. Suppose that g has asymptotics:

$$g(t) \sim \sum_{j=0}^{\infty} a_j t^{rj},$$

as $t \to 0^+$, for some $r > 0$. Then

$$\int_0^\infty t^\alpha g(t) e^{-xt}\, dt \sim \sum_{j=0}^{\infty} \frac{a_j \Gamma(\alpha + rj + 1)}{x^{\alpha+rj+1}},$$

as $x \to \infty$.

Proof: By definition, $\exists R > 0$ such that

$$g(t) = a_0 + a_1 t^r + ... + a_N t^{rN} + \mathrm{Rem}_N(t),$$

where $|\mathrm{Rem}_N(t)| = o(t^{rN})$ as $t \to 0^+$. Now split up the integral into three parts:

$$I(x) = \underbrace{\sum_{j=0}^{N} a_j \int_0^\infty t^{\alpha+rj} e^{-xt}\, dt}_{\text{(a) - first } N \text{ terms}} + \underbrace{\int_0^R \mathrm{Rem}_N(t) t^\alpha e^{-xt}\, dt}_{\text{(b) - can bound, since } |\mathrm{Rem}_N(t)| = o(t^{rN})}$$

$$+ \underbrace{\int_R^\infty \left(g(t) - \sum_{j=0}^{N} a_j t^{rj}\right) t^\alpha e^{-xt}\, dt}_{\text{(c) - need small}}.$$

Term (a) is just a sum of gamma-style integrals. They can be evaluated to get:

$$\sum_{j=0}^{N} a_j \int_0^\infty t^{\alpha+rj} e^{-xt}\, dt = \sum_{j=0}^{N} \frac{a_j \Gamma(\alpha + rj + 1)}{x^{\alpha+rj+1}}.$$

Now consider (c). By the growth assumption on g, we have $|g(t)t^\alpha e^{-xt}| \le Ke^{\beta t + \alpha \log(t) - xt}$. As $x \to \infty$, we can say $Ke^{\beta t + \alpha \log(t) - xt} \le Ke^{-\frac{1}{2}xt}$, say, if x sufficiently large. Similarly, we do the same for the sum:

$$\left|\sum_{j=0}^{N} a_j t^{rj}\right| |t^\alpha e^{-xt}| \le De^{-\frac{1}{2}xt},$$

for some D, for x sufficiently large. This bounds term (c) as

$$\int_R^\infty (D + K)e^{-\frac{1}{2}xt}\, dt = \frac{2(D + K)e^{-\frac{1}{2}xR}}{x} = o(x^{-N} e^{-\frac{1}{2}xR}),$$

so these terms are sub-dominant. Finally term (b) can be bounded:

$$\int_0^R \mathrm{Rem}_N(t) t^\alpha e^{-xt}\, dt \le \int_0^R Ct^{r(N+1)+\alpha} e^{-xt}\, dt$$

$$\le C \int_0^\infty t^{r(N+1)+\alpha} e^{-xt}\, dt = o(x^{-\alpha - r(N+1) - 1}),$$

by evaluating this gamma-style integral. □

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4 Laplace's method

4.1 Theory

Principle (Laplace localisation): The asymptotic expansion of

$$\int_a^b f(t) e^{x\phi(t)}\, dt$$

as $x \to \infty$ is determined entirely by contributions of arbitrarily small neighbourhoods of the points $\{t_\mu\}$ at which ϕ attains its maximum value.

General procedure:

1. Draw a sketch of $\phi(t)$ and determine where it attains its maxima, $\{t_\mu\}$.
2. State that the contribution from t_μ is the same as in an arbitrarily small neighbourhood of t_μ, and consider the integral from $-\epsilon + t_\mu$ to $\epsilon + t_\mu$ (or variants if near end-points).
3. Change variables so that we are expanding in powers of x. This allows us to see which terms are most important in the expansion.
4. Series expand near t_μ.
5. Extend ϵ to ∞, and integrate.

4.2 Examples

Example 1: Consider

$$I(x) = \int_0^{3\pi/2} e^{x\sin^2(t)}\, dt.$$

The maxima of $\sin^2(t)$ occur at $\pi/2$ and $3\pi/2$, as can be seen from a quick sketch. Near $t = \pi/2$, the contribution is:

$$\int_{\pi/2-\epsilon}^{\pi/2+\epsilon} e^{x\sin^2(t)}\, dt = \int_{-\epsilon}^{+\epsilon} e^{x\sin^2(\pi/2+u)}\, du.$$

Now expand $\sin^2(\pi/2+u) \approx 1 - u^2 + \frac{1}{4}u^4 + ...$, so the integral becomes

$$e^x \int_{-\epsilon}^{+\epsilon} e^{-xu^2+xu^4+\dots}\, du.$$

Now let $v = \sqrt{x}u$, to remove the x dependence from the first term in the exponent. This helps to order the terms. We are left with (relabelling ϵ):

$$\frac{e^x}{\sqrt{x}} \int_{-\epsilon}^{+\epsilon} e^{-v^2+\frac{1}{x}v^4+\dots}\, du = \frac{e^x}{\sqrt{x}} \int_{-\epsilon}^{+\epsilon} e^{-v^2}\left(1 + \frac{1}{x}v^4 + o(x^{-2})\right) du.$$

Now extend $\epsilon \to \infty$ and evaluate. A similar method applies at the other maximum.

Example 2: Consider

$$\int_0^1 e^{x(2t^2-1)^2}\left(\frac{\sin(t)}{t(1-t)}\right) dt.$$

We have $\phi(t) = (2t^2-1)^2$, which has maxima at $t = 0$ and $t = 1$. Taylor expansion shows that f is singular at $t = 1$ which magnifies the contribution from this maximum. Writing $u = 1 - t$, we have:

$$f(t) \sim u^{-1}\sin(1) + u(\sin(1) - \cos(1)) + ...$$

Using this expansion, the contribution at $t = 1$ is given by:

$$\int 1 - \epsilon^1 e^{x\phi(t)} f(t)\, dt$$

$$= \int_0^{\epsilon} e^{x(2(1-u)^2-1)^2}(u^{-1}\sin(1) + u(\sin(1) - \cos(1)) + ...)\, du.$$

Changing variables to remove x from the lowest power of u in the exponent, we can then just continue as usual.

4.3 The general case

Higher order maxima: Consider

$$\int_a^b f(t)e^{x\phi(t)}\, dt,$$

and suppose that $\phi(t)$ has a maximum at $c \in (a, b)$ with $\phi'(c) = 0$, $\phi''(c) = 0$, ... $\phi^{(s-1)}(c) = 0$, and $\phi^{(s)} < 0$. Suppose f is smooth at c. Carrying out the local expansion leads to an integral of the form

$$e^{x\phi(c)} \int_{c-\epsilon}^{c+\epsilon} f(t)e^{-x|\phi^{(s)}(t)|\cdot(t-c)^s/s!+\dots}\, dt,$$

which leads to a contribution (by Taylor expanding $f(t)$, making substitution to make a gamma integral):

$$\frac{2e^{x\phi(c)}f(c)\Gamma(1/s)(s!)^{1/s}}{sx^{1/s}|\phi^{(s)}(c)|^{1/s}}.$$

Slogan: A maximum of order s typically contributes a term $e^{x\phi(c)}/x^{1/s}$ to the asymptotics; that is, *the flatter it is, the more it contributes*.

Endpoints: At an endpoint a or b, we generically have $\phi'(a) \neq 0$, $\phi'(b) \neq 0$. If there is a maximum at a, and f is smooth there, the contribution is:

$$\int_a^{a+\epsilon} f(t)e^{x\phi(t)}\, dt \sim \frac{-f(a)e^{x\phi(a)}}{x\phi'(a)}.$$

Slogan: Endpoint maxima typically contribute terms of the form $e^{x\phi(a)}/x$.

Singular f: The above analysis does not apply at points where f is singular. The presence of singularities in f at maxima magnifies the contribution, which can be seen from Watson's Lemma:

$$g(t) \sim \sum a_j t^{rj} \quad \Rightarrow \quad \int t^\alpha g(t)e^{-xt}\, dt \sim \sum \frac{a_j\Gamma(\alpha + rj + 1)}{x^{\alpha+rj+1}}.$$

4.4 Optimal truncation

To actually calculate with an asymptotic expansion, we use the *optimal truncation rule*: keep all the terms in the expansion up to the one **before the smallest**.

5 Oscillatory integrals

5.1 A useful integral

Theorem: We have:

$$\int_0^\infty t^\gamma e^{ixt^p}\,dt = \frac{1}{p}\left(\frac{1}{x}\right)^{\frac{\gamma+1}{p}}\Gamma\left(\frac{\gamma+1}{p}\right)\exp\left(\frac{i\pi(\gamma+1)}{2p}\right),$$

provided $x > 0$, $-1 < \gamma < 0$ and $p \geq 1$. Taking the complex conjugate of the whole equation also gives another useful integral.

Proof: First change variables to $u = t^p$. Then use a quarter-circle contour integral in the first quadrant. □

5.2 The method of stationary phase

Principle (Stationary phase): The asymptotic expansion of the integral

$$I(x) = \int_a^b f(t)e^{ix\phi(t)}\,dt$$

is completely determined by the points of *stationary phase*, i.e. points t_μ for which $\phi'(t_\mu) = 0$, and endpoints a, b.

5.3 Stokes' problem

Application of the method of stationary phase is almost identical to that of Laplace integrals. Consider the example:

$$\int_0^\infty \cos(x(t^3 - t))\,dt = \frac{1}{2}\int_{-\infty}^\infty e^{ix(t^3-t)}\,dt,$$

as $x \to \infty$ (sine is odd so cancels, so really equality above). We have $\phi(t) = t^3 - t$, so the points of stationary phase are $t = \pm 1/\sqrt{3}$. The method then proceeds the same as when studying Laplace integrals.

5.4 The Riemann-Lebesgue Lemma

Often to justify use of the method of stationary phase (or more likely, integration by parts used on a stationary phase-type integral), we need:

Theorem (Riemann-Lebesgue Lemma): If F is absolutely integrable, we have:

$$\lim_{\omega\to\infty}\int_{-\infty}^\infty F(t)e^{i\omega t}\,dt = 0.$$

Proof: Non-examinable. □

5.5 Endpoint contributions

From an endpoint for an oscillatory integral, we generically expect a contribution $O(e^{ix\phi(a)}/x)$, for exactly the same reason as Laplace integrals. Again this can be magnified by the presence of singularities.

Often integration by parts can be the easiest way to proceed near endpoints.

6 The method of steepest descent

6.1 Description of method

We now consider integrals of the form

$$I(k) = \int_C f(z)e^{kh(z)}\,dz.$$

where C is a contour in the complex plane, and $f(z)$, $h(z)$ are holomorphic functions, as $k \to \infty$.

Procedure:

1. Find the *saddle points*, i.e. points where $h'(z) = 0$.
2. Find the paths of *steepest ascent/descent* through the saddle points and the end points, i.e. those paths where the imaginary part of $h(z)$ is constant. So if z_μ is a saddle point or an end point, the paths of steepest ascent/descent through z_μ are $\mathrm{Im}(h(z)) = \mathrm{Im}(h(z_\mu))$. The best way to do this is to write $z = x + iy$ or $z = re^{i\theta}$.
3. Decide which paths are ascent paths and which are descent paths. This can be achieved by considering $\mathrm{Re}(h(z))$ along the steepest ascent/descent paths.
4. Sketch the paths in the complex plane, along with the original contour C. Decide how we can deform C so that it passes along steepest ascent/descent paths as appropriate.
5. Linearise the paths going through the saddle points and endpoints. Then proceed to use Laplace's method to find the asymptotic contributions at each of these points.

6.2 Example of steepest descent

Consider

$$\int_0^1 e^{ixt^3}\,dt,$$

for $x \to \infty$. First we find the saddle points: $h(t) = it^3 \Rightarrow h'(t) = 3it^2$, so only saddle point is at 0 which is also an end point.

Now we find the paths of steepest descent/ascent through the end points and the saddle points. The paths are given by $\mathrm{Im}(h(t)) = \mathrm{Im}(h(0))$ and $\mathrm{Im}(h(t)) = \mathrm{Im}(h(1))$. In the first case it is best to write $z = re^{i\theta}$, whence the paths are $r^3\cos(3\theta) = 0 \Rightarrow \cos(3\theta) = 0$. In the second case, we linearise to get paths looking like $y \approx \pm\sqrt{3}x$. Considering the real part gives the paths of steepest descent, which we integrate along instead.

6.3 Asymptotics of the Airy functions

Airy's equation is given by $y'' - xy = 0$. By using the kernel method for ODEs (see FCM) we can derive the integral representations of the solutions. Deforming the contours of integration of these solutions appropriately, we find the two independent solutions of Airy's equation:

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(\frac{s^3}{3}+xs\right)\,ds,$$

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^{\infty}\left(\exp\left(xt-\frac{t^3}{3}\right)+\sin\left(xt+\frac{t^3}{3}\right)\right)\,dt.$$

Using the method of steepest descent, it is possible to derive the asymptotics of these functions as $x \to \infty$ and $x \to -\infty$.

7 The Liouville-Green/WKB method

7.1 Liouville-Green approximations

We study equations of the form

$$\epsilon^2 y'' = Q(x)y(x)$$

where $\epsilon \ll 1$. A general second-order equation with a y' term can be reduced to this form by the substitution $y(x) = u(x)v(x)$ and choosing v so that u' terms vanish.

Method: We aim to find solutions which look (formally) like $\exp\left(\frac{1}{\epsilon}S_0 + S_1 + \epsilon S_2 + ...\right)$ by substituting this form directly into the equation and equating coefficients of powers of ϵ. In general, this allows us to derive the *Liouville-Green approximate solutions*, by retaining terms up to ϵ^1 when we substitute our guess in. For $Q(x) < 0$, we have:

$$y(x) \approx |Q(x)|^{-1/4}\Bigg(A_+\exp\left(\frac{i}{\epsilon}\int^x |Q|^{1/2}\right) + A_-\exp\left(\frac{-i}{\epsilon}\int^x |Q|^{1/2}\right)\Bigg)$$

and for $Q(x) > 0$, we have:

$$y(x) \approx |Q(x)|^{-1/4}\Bigg(a_+\exp\left(\frac{1}{\epsilon}\int^x |Q|^{1/2}\right) + a_-\exp\left(\frac{-1}{\epsilon}\int^x |Q|^{1/2}\right)\Bigg)$$

7.2 Eigenvalue problems

An application of this method is to find eigenvalues. Consider $\epsilon^2 y'' = Q(x)y$ with $y(0) = y(1)$. Assume $Q(x) < 0$. Then

$$y(x) \approx |Q(x)|^{-1/4}\Bigg(\alpha_+\cos\left(\frac{1}{\epsilon}\int_0^x |Q|^{1/2}\right) + \alpha_-\sin\left(\frac{1}{\epsilon}\int_0^x |Q|^{1/2}\right)\Bigg)$$

by changing the constants in the above. The condition $y(0) = 0$ forces $\alpha_+ = 0$, and the condition $y(1) = 0$ gives the spectrum:

$$\sin\left(\frac{1}{\epsilon}\int_0^1 |Q(x)|^{1/2}\right) = 0 \quad\Rightarrow\quad \epsilon_n = \frac{1}{n\pi}\int_0^1 |Q(x)|^{1/2}.$$

7.3 The connection problem

We have two separate formulae for $Q(x) > 0$ and $Q(x) < 0$, but what happens when we cross a point where $Q(x) = 0$? Rewrite $q(x) = -Q(x)$ so that we are studying $\epsilon^2 y'' + q(x)y = 0$. Suppose that $q(x) > 0$ for $x < 0$ and $q(x) < 0$ for $x > 0$. Suppose $q(0) = 0$ (see the diagram).

Away from the origin, the following Liouville-Green approximations hold:

$$y(x) = |q|^{-\frac{1}{4}} \left(a_+ \exp\left(\int^x |q|^{\frac{1}{2}}/\epsilon \right) + a_- \exp\left(-\int^x |q|^{\frac{1}{2}}/\epsilon \right) \right),$$

for $x < 0$ and

$$y(x) = |q|^{-\frac{1}{4}} \left(D \cos\left(\int^x |q|^{\frac{1}{2}}/\epsilon + \frac{\pi}{4} \right) + E \sin\left(\int^x |q|^{\frac{1}{2}}/\epsilon + \frac{\pi}{4} \right) \right),$$

for $x > 0$.

Now, in a neighbourhood of the origin, suppose that $q(x) \approx q(0) - kx$, where $k > 0$. Then the equation is approximated by $\epsilon^2 y'' = kxy$, i.e. a scaled version of the standard Airy equation. By substituting $u = \left(\frac{k}{\epsilon^2}\right)^{1/3} x$, it is clear that the general solution is:

$$y(x) = \alpha \mathrm{Ai}\left(\left(\frac{k}{\epsilon^2}\right)^{1/3} x \right) + \beta \mathrm{Bi}\left(\left(\frac{k}{\epsilon^2}\right)^{1/3} x \right).$$

Now we hope that large asymptotics (in particular those in the region $\epsilon^{2/3} \ll |x| \ll 1$, where $|x|/\epsilon^{2/3} \gg 1$) should match with the Liouville-Green solutions in the other regions. Using the asymptotics for the Airy function, we have:

$$y(x) \sim \frac{\epsilon^{1/6}\alpha}{2\sqrt{\pi}k^{1/12}x^{1/4}} \exp\left(\frac{-2}{3} \left(\frac{k}{\epsilon^2}\right)^{1/2} x^{3/2} \right) + \frac{\epsilon^{1/6}\beta}{\sqrt{\pi}k^{1/12}x^{1/4}} \exp\left(\frac{2}{3} \left(\frac{k}{\epsilon^2}\right)^{1/2} x^{3/2} \right),$$

when $x > 0$, and

$$y(x) \sim \frac{\epsilon^{1/6}\alpha}{2\sqrt{\pi}k^{1/12}|x|^{1/4}} \sin\left(\frac{-2}{3} \left(\frac{k}{\epsilon^2}\right)^{1/2} |x|^{3/2} + \frac{\pi}{4} \right) + \frac{\epsilon^{1/6}\beta}{\sqrt{\pi}k^{1/12}|x|^{1/4}} \cos\left(\frac{2}{3} \left(\frac{k}{\epsilon^2}\right)^{1/2} |x|^{3/2} + \frac{\pi}{4} \right),$$

when $x < 0$. Putting $q = -kx$ in the Liouville-Green solutions in the other regions and comparing coefficients, we arrive at the *connection formulae*:

$$a_+ = E, \qquad 2a_- = D.$$

7.4 The WKB approximation

The WKB approximation is used to approximate the energy levels of a quantum system. Consider the problem:

$$-\frac{\hbar^2}{2m} y'' + Vy = Ey.$$

Rewrite as $\hbar^2 y'' = -2m(E - V)y$ and treat $\hbar$ as the small parameter. There are two *turning points* A and B (which are points when $Q(x) = 0$), occuring when $V(x) = E$ (see diagram).

For $x > B$ we use Liouville-Green, but are forced to pick the decaying solution:

$$y(x) = \frac{c_1}{|2m(E - V)|^{1/4}} \exp\left(-\frac{1}{\hbar} \int_B^x (2m(E - V))^{1/2}\, dx \right).$$

By the connection formulae, this forces the solution in $A < x < B$ to be:

$$y(x) = \frac{2c_1}{|2m(E - V)|^{1/4}} \sin\left(\frac{1}{\hbar} \int_x^B (2m(E - V))^{1/2}\, dx + \frac{\pi}{4} \right).$$

Similarly for $x < A$ we need a decaying Liouville-Green solution, which by the connection formulae forces the solution in $A < x < B$ to be:

$$y(x) = \frac{2c_2}{|2m(E - V)|^{1/4}} \sin\left(\frac{1}{\hbar} \int_A^x (2m(E - V))^{1/2}\, dx + \frac{\pi}{4} \right).$$

Thus in the region, we have that these two solutions are proportional, and so:

$$\sin\left(\frac{1}{\hbar}\int_A^x (2m(E-V))^{1/2}\,dx + \frac{\pi}{4}\right) \propto$$
$$\sin\left(\frac{1}{\hbar}\int_x^B (2m(E-V))^{1/2}\,dx + \frac{\pi}{4}\right).$$

Rewriting the right hand side, we have:

$$\sin\left(\frac{1}{\hbar}\int_x^B (2m(E-V))^{1/2}\,dx + \frac{\pi}{4}\right) =$$
$$-\sin\left(\frac{1}{\hbar}\int_B^x (2m(E-V))^{1/2}\,dx - \frac{\pi}{4}\right) =$$

$$-\sin\left(\frac{1}{\hbar}\int_A^x (2m(E-V))^{1/2}\,dx +\right.$$
$$\left.\frac{\pi}{4} - \left(\frac{1}{\hbar}\int_A^B |2m(E-V)|^{1/2}\,dx + \frac{\pi}{2}\right)\right),$$

from which it follows that

$$\left(n+\frac{1}{2}\right)\pi\hbar = \int_A^B \sqrt{2m(E_n - V(x))}\,dx,$$

for $n \in \mathbb{Z}$. This is the *Bohr-Sommerfeld quantisation condition*.