# Part III: Classical and Quantum Solitons - Revision

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## 1 Kinks

### 1.1 Definitions

**Convention:** The Minkowski metric in this course is mostly negative.

**Definition:** The *Lagrangian* of a theory in $1+1$ dimensions is

$$L = \int_{-\infty}^{\infty} \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right) \, dx.$$

We can separate this into *kinetic* and *potential energy* via:

$$T = \int_{-\infty}^{\infty} \frac{1}{2} \dot{\phi}^2 \, dx, \qquad V = \int_{-\infty}^{\infty} \left( \frac{1}{2} \phi'^2 + U(\phi) \right) \, dx.$$

The *action* is the time integral of the Lagrangian. As usual, extremising the action gives the *Euler-Lagrange equations*:

$$\partial_\mu \partial^\mu \phi + \frac{dU}{d\phi} = 0.$$

**Definition:** A *static solution* $\phi(x,t) = \phi(x)$ is one which is time independent, i.e. solves

$$\frac{d^2\phi}{dx^2} = \frac{dU}{d\phi}.$$

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### 1.2 The Bogomolny equations

**Definition:** A *kink* is a particle-like solution to the field equation connecting two vacua.

To generate kinks, we minimise the *energy*.

**Theorem:** Minimising the energy

$$E = \int_{-\infty}^{\infty} \left( \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\phi')^2 + U(\phi) \right) \, dx$$

gives the *Bogomolny equations* $\phi' = \pm \frac{dW}{d\phi}$, where $W$ is defined through $U(\phi) = \frac{1}{2} \left( \frac{dW}{d\phi} \right)^2$. The minimum values of the energy are $\pm(W(\phi(\infty)) - W(\phi(-\infty)))$ respectively.

*Remark:* Clearly the energy is positive. Thus we must consider both signs throughout, but pick only the one giving a positive energy as a physical solution.

*Proof:* Complete the square in $E$. This is called the *Bogomolny rearrangement*. With the given $W$ definition, the energy is:

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \left( \phi'^2 + \left( \frac{dW}{d\phi} \right)^2 \right) \, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left( \phi' \mp \frac{dW}{d\phi} \right)^2 \, dx \pm \int_{-\infty}^{\infty} \frac{dW}{d\phi} \frac{d\phi}{dx} dx.$$

Doing the second integral immediately gives the result. $\square$

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**Theorem:** A solution of the Bogomolny equations is a solution of the field equations.

*Proof:* Simply note

$$\frac{d^2\phi}{dx^2} = \pm \frac{d}{dx} \left( \frac{dW}{d\phi} \right) = \pm \frac{d\phi}{dx} \left( \frac{d^2W}{d\phi^2} \right) = \frac{dW}{d\phi} \left( \frac{d^2W}{d\phi^2} \right) = \frac{dU}{d\phi}. \square$$

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**Theorem:** A kink may only connect adjacent vacua.

*Proof:* Restrict to the case of quadratic vacua. Near a quadratic vacuum $\phi_0$, we have

$$U(\phi) \approx U(\phi_0) + \frac{1}{2} (\phi - \phi_0)^2 U''(\phi_0).$$

where $U''(\phi_0) > 0$. The field equation $\phi'' = U'(\phi)$ then gives

$$\phi''(x) = (\phi - \phi_0) U''(\phi_0).$$

Integrating, we get

$$\phi(x) = \phi_0 + C_1 e^{x\sqrt{U''(\phi_0)}} + C_2 e^{-x\sqrt{U''(\phi_0)}}.$$

For any finite $x$, we have $\phi \neq \phi_0$. It follows that $\phi$ can only equal a vacuum value at $x = \pm\infty$, and hence can connect only adjacent vacua ($\phi$ cannot pass through another vacuum value at some finite $x$). $\square$

### 1.3 $\phi^4$ kinks

**Example:** Consider normalised $\phi^4$ theory, i.e. $U(\phi) = \frac{1}{2}(1-\phi^2)^2$. There are two vacua (i.e. minima of the potential) at $\pm 1$; consider a kink that goes from $-1$ at $x=-\infty$ to $+1$ at $x=\infty$. This gives

$$W(\phi) = \phi - \frac{1}{3}\phi^3.$$

Our choice of sign means $W(\phi(\infty)) = W(1) = 2/3$ and $W(\phi(-\infty)) = W(-1) = -2/3$, so that $E = 4/3$ is the energy (or *mass* by energy-mass equivalence) of the kink. Hence we chose the right sign for $W$.

Since we want a kink with $\phi' > 0$ (i.e. increasing from one vacuum to the next), the Bogomolny equation for the kink's shape is

$$\frac{d\phi}{dx} = 1 - \phi^2,$$

which has solution $\phi(x) = \tanh(x-a)$, where $a$ is a constant of integration. There's a clear symmetry $\phi \mapsto -\phi$, $x - a \mapsto a - x$ about $a$, implying that $a$ is the centre of the kink (this is harder to define when there is no symmetry).

The *anti-kink* is a separate solution linking the vacua from $1$ at $x=-\infty$ to $+1$ at $x=\infty$. This has the opposite sign in the Bogomolny equations since we want the kink to be decreasing. The solution is $\phi(x) = -\tanh(x-a)$. It also has energy $4/3$.

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**Definition:** The parameter $a$ is called the *modulus* or *collective coordinate* of the kink solution. The *moduli space* of the solution is the set of all possible $a$, which in this case is $\mathbb{R}$.

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### 1.4 Derrick's Theorem

**Derrick's Theorem:** Work in $d+1$ dimensions, and suppose our field theory has a kink solution. Then if $T$ is the kinetic energy and $V$ is the potential energy, we have

$$(2-d)T - dV = 0.$$

*Proof:* Let $\phi(x)$ be a kink solution minimising $E$. Consider rescaling $\phi(x) \mapsto \phi(\lambda x)$. Then

$$E = T + V \mapsto E'(\lambda) = \lambda^{2-d}T + \lambda^{-d}V,$$

by a trivial calculation. But $\phi(x)$ minimises $E$, hence we must have a minimum of $E'(\lambda)$ when $\lambda = 1$. Differentiating and setting $\lambda = 1$ gives the required condition. □

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In $d = 1$ dimensions, Derrick's Theorem reduces to $T = V$, i.e.

$$\frac{1}{2}\int_{-\infty}^{\infty} \dot{\phi}^2 \, dx = \frac{1}{2}\int_{-\infty}^{\infty} {\phi'}^2 \, dx.$$

In particular, the energy/mass of the kink is $M = E = \frac{1}{2}T = \frac{1}{2}V$ in $1+1$ dimensions.

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Another use of Derrick's Theorem is ruling out soliton solutions. In $d = 2$ dimensions, the Theorem gives $V = 0$, which tells us the potential energy of any soliton solution in $2$ dimensions must be zero.

In $d = 3$ dimensions, we find $-T = 3V$. Since $T$, $V$ are both positive, this implies $T = V = 0$. So there are no soliton solutions. Similarly for $d \geq 4$.

*However* we can evade Derrick's Theorem by introducing additional fields with their own energy terms that have different scaling behaviours. When we study vortices we introduce a gauge field kinetic term, which allows solitons with non-zero potential energy in $2+1$ dimensions.

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### 1.5 The moduli space approximation

Let's now consider adding dynamics to the theory. Since the theory is Lorentz invariant, we need only Lorentz boost:

$$\phi(x,t) = \tanh(\gamma(x - vt)),$$

where $\gamma = (1-v^2)^{-1/2}$ as usual.

It's sometimes useful to view this *adiabatically*, that is for $v \ll 1$. In general, we can achieve this by making the modulus *time-dependent*. This is called the *moduli space approximation* to the motion.

**Theorem:** For an approximate dynamic kink $\phi(x,t) = \tanh(x - a(t))$, where $a(t)$ is a dynamic modulus, we have $a(t) = vt$.

*Proof:* Note we have

$$\dot{\phi} = -\frac{da}{dt}\phi'.$$

Substitute this into the kinetic and potential energies to find:

$$T = \frac{1}{2}M\left(\frac{da}{dt}\right)^2, \quad V = M.$$

where $M = 4/3$ is the mass (energy) of the kink (note we've used Derrick's Theorem here to evaluate the integral of ${\phi'}^2$ in the kinetic and potential energies). Thus the effective Lagrangian describing the kink's motion is

$$L = \frac{1}{2}M\dot{a}^2 - M.$$

This is why $a$ is called a collective coordinate. It reduced all field dynamics to a $1$D particle problem. The equation of motion of this Lagrangian is $M\ddot{a} = 0$, which gives $a = vt$ (plus arbitrary constant relating to starting position). □

**Theorem:** The momentum of a $\phi^4$ kink in the moduli space approximation is $P = M\dot{a}$.

*Proof:* Computing the energy momentum tensor of the theory, we find:

$$P = -\int_{-\infty}^{\infty} \dot{\phi}\phi' \, dx.$$

In the moduli space approximation, $\dot{\phi} = -\phi'\dot{a}$ as above, and hence doing the integral over $\phi'^2$ to get mass $M$ (as per Derrick's Theorem), we get the result. □

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## 1.6 Quantisation of kinks

Quantisation is easy in the moduli space approximation. The Lagrangian is $L = \frac{1}{2}M\dot{a}^2$ as we calculated before, so the Hamiltonian is

$$H = P\dot{a} - L = \frac{P^2}{2M}.$$

Quantising simply gives $P \mapsto -i\hbar\partial_a$, i.e.

$$H = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial a^2}$$

Stationary states are of the form $\psi(x) = e^{i\kappa a}$, with eigenvalues $P = \hbar\kappa$, $H = \hbar^2\kappa^2/(2M)$.

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Quantisation of the full field theory is very difficult. We end up seeing two particles: *kinks* and *mesons*. Mesons are quantised waves around the vacuum. The mesons also interact with the kink.

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## 1.7 $\phi^6$ kinks

**Example:** Let $U(\phi) = \frac{1}{2}(1-\phi^2)\phi^2$. There are now three vacua, but we know we can only connect adjacent vacua, so there are two types of kinks (and two types of anti-kinks).

For the kink from $\phi = 0$ to $\phi = 1$, the suitable root for $W$ is $W(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4$. The Bogomolny equation is

$$\frac{d\phi}{dx} = (1-\phi^2)\phi,$$

which gives solution $\phi(x) = (1 + e^{-2(x-a)})^{-1/2}$. The energy/mass of the kink is $1/4$.

Notice this kink is *antisymmetric* because the vacua have slightly different local behaviour. $\phi = \pm 1$ are symmetric vacua, but $\phi = 0$ and $\phi = 1$ are not.

## 1.8 The sine-Gordon soliton

**Example:** Sine-Gordon theory has $U(\phi) = 1 - \cos(\beta\phi)$. This has infinitely many vacua, but we can only connect adjacent ones. By periodicity, all vacua are the same and thus the kinks are symmetric about their centres.

Solving the Bogomolny equations in this instance gives

$$\phi = \frac{4}{\beta}\arctan\left(\exp\left(\pm\beta(x-a)\right)\right).$$

(Note we must integrate $\operatorname{cosec}(x)$, which has integral $-\log(\cot(\frac{1}{2}x))$.)

The kink solution connecting the vacua at $\phi = 0$, $x = -\infty$ and $\phi = 2\pi/\beta$, $x = \infty$ turns out to give the plus sign. The energy of the kink is $8/\beta$.

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Restrict to the case $\beta = 1$. Since $\phi$ is an angular coordinate in sine-Gordon theory, it is useful to *identify* $\phi \sim \phi + 2\pi$. We then think of $\phi$ as a compact variable $\phi : \mathbb{R} \to S^1$, i.e. a map into the circle.

There is now a single vacuum at $\phi = 0$ modulo $2\pi$, which means our boundary conditions become $\phi = 0$ when $x \to \pm\infty$. Since we have boundary conditions which are periodic at $x = \pm\infty$, we may identify $\phi$ as a map:

$$\phi : S^1_\infty \to S^1,$$

where $S^1_\infty$ is the compactification of $\mathbb{R}$, identifying $x = \pm\infty$.

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## 1.9 Topology of the sine-Gordon soliton

Now $\phi : S^1_\infty \to S^1$ is a map from a circle to a circle. As $\phi$ goes once around $S^1_\infty$, how many times does it go round $S^1$?

**Definition:** If $\phi(-\infty) = 2n_L\pi$ (must be an integer multiple of $2\pi$ since $\phi = 0$ modulo $2\pi$ at $-\infty$) and we go round the infinite circle once. Suppose we arrive at $\phi(\infty) = 2n_R\pi$. Then the *winding number* is

$$Q = n_R - n_L.$$

**Example:** For the soliton solution above joining $0$ to $2\pi$, the winding number is $1$.

We should think of a winding number $Q = n$ as the superposition of $n$ kinks, as shown below.

For a single anti-kink, $Q = -1$. So we should interpret $Q$ as

$$Q = \text{number of kinks} - \text{number of anti-kinks}$$

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We can also obtain $Q$ from Noether's Theorem.

**Theorem:** The current

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi$$

is conserved, and its associated charge is $Q$.

*Proof:* We have:

$$\partial_\mu j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\mu\partial_\nu\phi = 0.$$

By Noether's Theorem, the associated charge is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\partial_x\phi\, dx = \frac{1}{2\pi}(\phi(\infty) - \phi(-\infty)) = Q. \quad \Box$$

This current does *not* come from a symmetry of the theory, and hence it is strange that it is conserved with non-trivial consequences (e.g. $(1,2)$ is clearly a vector that is conserved, but this tells us nothing). It's thus called a *topological current*.

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Finally, we can obtain $Q$ through a differential geometry perspective. Consider $\phi : S^1_\infty \to S^1$. Both spaces are compact, and are of the same dimension, so there is a notion of the *degree* of the map (see later).

**Definition:** The *degree* of this map is the integral over $S^1_\infty$ of the pullback of any normalised volume form on $S^1$.

**Theorem:** The degree of $\phi$ is $Q$, the winding number.

*Proof:* A normalised volume form on $S^1$ is $\frac{1}{2\pi}d\phi$. The pullback is just like 'changing coordinates':

$$\frac{1}{2\pi}d\phi = \frac{1}{2\pi}\frac{d\phi}{dx}dx.$$

Integrating this from $-\infty$ to $\infty$, we get precisely the earlier characterisation of $Q$. $\Box$

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### 1.10 The dynamic sine-Gordon kink

**Example:** The $2$-kink dynamical solution to the sine-Gordon equation is

$$\phi(x,t) = 4\arctan\left(\frac{v\sinh(\gamma x)}{\cosh(\gamma v t)}\right).$$

We can sketch this to see it is a pair of kinks that come in from $\infty$, approach and then repel, and return to $\infty$.

To establish the distance of closest approach, write the solution as

$$\begin{aligned}\phi(x,t) &= 4\arctan\left(e^{\gamma(x-a(t))} - e^{-\gamma(x+a(t))}\right)\\ &= 4\arctan(e^{\gamma(x-a(t))}) - 4\arctan(e^{-\gamma(x+a(t))}).\end{aligned}$$

where

$$a(t) = \frac{1}{\gamma}\log\left(\frac{2}{v}\cosh(\gamma v t)\right).$$

We've also used

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right).$$

It's now clear that this is two kinks together, and we can identify their centres as $\pm a(t)$. Thus they are closest when

$$2a(t) = \frac{2}{\gamma}\log\left(\frac{2}{v}\cosh(\gamma v t)\right)$$

is minimised. Hyperbolic cosine is minimised when $t = 0$. Assuming small $v$, we have closest approach distance $2\log(2/v)$.

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## 2 Vortices

### 2.1 The Abelian-Higgs model

**Definition:** The *Abelian-Higgs model* has Lagrangian

$$\mathcal{L} = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*D^\mu\phi - \frac{\lambda}{8}(1-\phi^*\phi)^2,$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ for some gauge field $a_\mu$ and some scalar field $\phi$. The covariant derivative is $D_\mu\phi = \partial_\mu\phi - ia_\mu\phi$. The theory is $U(1)$ gauge invariant under the infinitesimal transformations

$$\phi \mapsto \phi + i\alpha\phi, \quad a_\mu \mapsto a_\mu + \partial_\mu\alpha,$$

or alternatively the finite transformations:

$$\phi \mapsto U\phi, \quad a_\mu \mapsto Ua_\mu U^{-1} - i(\partial_\mu U)U^{-1}.$$

We think of $a$ as a one form and $f_{\mu\nu}$ as the components of a two form:

$$f = da = (\partial_1 a_2 - \partial_2 a_1)dx^1 \wedge dx^2 = f_{12}dx^1 \wedge dx^2.$$

**Definition:** $B = f_{12}$ is the *magnetic field.*

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**Theorem:** The field equations are:

$$\partial_\mu f^{\mu\nu} = \frac{1}{2}i((D^\nu\phi)^*\phi - \phi^* D^\nu\phi),$$
$$D_\mu D^\mu\phi = \frac{1}{2}\lambda(1-\phi^*\phi)\phi.$$

*Proof:* Quick calculation. $\Box$

The Abelian Higgs model is relevant as a description of superconductors. If $\lambda > 1$, we get *Type II behaviour*, where vortices repel one another. They form a *vortex lattice*.

If $\lambda < 1$, we get *Type I behaviour*, where vortices attract. They coalesce to form one giant structure.

We will be interested in *critical coupling* at $\lambda = 1$.

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## 2.2 Transformation to polars

It will be useful to work in polar coordinates. Use $x^1 = r\cos(\theta)$ and $x^2 = r\sin(\theta)$, implying:

$$dx^1 = \cos(\theta)dr - r\sin(\theta)d\theta, \quad dx^2 = \sin(\theta)dr + r\cos(\theta)d\theta.$$

**Theorem:** In polar coordinates, $r = \sqrt{{x^1}^2 + {x^2}^2}$ and $\theta = \arctan(y/x)$, we have

$$a_r = \frac{a_1x^1 + a_2x^2}{r}, \quad a_\theta = -a_1x^2 + a_2x^1,$$
$$f_{r\theta} = rf_{12} = rB.$$

*Proof:* Write $a = a_1dx^1 + a_2dx^2$ formally, and expand in terms of the above. Also write $f = f_{r\theta}dr \wedge d\theta$ and expand in terms of the above. □

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## 2.3 Vortex topology

Before constructing explicit solutions, consider properties of vortices.

First, we ask what the boundary conditions are. As in the kink case, the boundary conditions should force us to tend to the vacuum solution at $\infty$. Since the energy is

$$E = \int d^2x \, \left(\frac{1}{2}B^2 + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) + \frac{\lambda}{8}(1-\phi^*\phi)^2\right).$$

to ensure $E \to 0$ asymptotically, we require $B \to 0$, $|\phi| \to 1$ and $D\phi \to 0$ as $r \to \infty$.

**Definition:** The *field at infinity* is written $\phi_\infty(\theta)$, and is defined by

$$\phi_\infty(\theta) = \lim_{r\to\infty} \phi(r,\theta).$$

We require $|\phi_\infty(\theta)| = 1$ by the above, so we write $\phi_\infty(\theta) = e^{i\chi(\theta)}$ where $\chi(\theta)$ is the *phase at infinity*.

We now see that $\chi$ is a map $\chi : S^1 \to S^1$, so as usual it has a winding number.

**Definition:** Let $\phi_\infty(0)$ be the value of the field at infinity at angle $0$. Rotate around to $\theta = 2\pi$ and consider $\phi_\infty(2\pi)$. Since the field is single-valued, we must have $\chi(\theta)$ changing only by $2\pi N$ for some integer $N$; we call $N$ the *winding number* of the field.

**Theorem:** The winding number has the following properties:

(i) $N$ is gauge-invariant;

(ii) $N$ is a topologically conserved quantity;

(iii) $N = \frac{1}{2\pi}\int_{\mathbb{R}^2} B \, d^2x$;

(iv) $N$ is the number of *isolated zeroes* of $\phi$ counted with multiplicity.

*Proof:* (i) Let $\phi \mapsto e^{i\alpha(r,\theta)}\phi$ be a gauge transformation. Then $\alpha(r,\theta)$ is a map from $S^1$ to $S^1_\infty$ for fixed $r$, hence it has some winding, say $N_\alpha(r)$, dependent on the fixed radius. Thus gauge transformations transform $N \mapsto N + N_\alpha(\infty)$.

Note that $N_\alpha$ is an integer, since it is a winding number, hence under continuous deformations it cannot change. Thus $N_\alpha(0) = N_\alpha(\infty)$. But clearly no winding on a circle which is just a point so $N_\alpha(0) = 0$.

(ii) $N$ is an integer so is invariant under continuous deformations of the fields, including time evolution.

(iii) We note that $D_\theta\phi \to 0$ at infinity. Hence $\partial_\theta\phi_\infty = ia_\theta^\infty\phi_\infty$. It follows from the definition of $\chi$ that $a_\theta^\infty = \partial_\theta\chi$.

Now transform to a gauge where $a_r = 0$ by making the gauge transformation:

$$a_r \mapsto a_r - \partial_r \int_0^r a_r(r,\theta)\, dr = a_r - a_r = 0,$$

i.e. take $\alpha = -\int_0^r a_r(r,\theta)\, dr$. Then

$$\int_{\mathbb{R}^2} B\, d^2x = \int_{\mathbb{R}^2} f = \int_0^\infty\int_0^{2\pi} f_{r\theta}\, drd\theta = \int_0^{2\pi} a_\theta^\infty\, d\theta = \int_0^\infty \partial_\theta\chi\, d\theta.$$

This is equal to $\chi(2\pi) - \chi(0) = 2\pi N$, and the result follows.

(iv) By diagram. On a circle around an isolated zero $\mathbf{X}_i$, $\phi$'s argument changes by $2\pi n_i$ where $n_i$ is called the *multiplicity* of the zero. Now draw a big circle with some smaller circles coming off it:

It's then clear that $N$ is the sum of the multiplicities of the isolated zeroes. □

## 2.4 The basic vortex

As we say in the above Theorem, we can always set $a_r = 0$. Assuming we have a circularly symmetric solution then, $a_\theta = f(r)$ and $\phi = h(r)e^{i\theta}$.

This ansatz implies that the phase at infinity is $\chi(\theta) = \theta$. Hence using $a_\theta = \partial_\theta \chi$, we get the boundary condition $f(\infty) = 1$. We also use the boundary condition $f(0) = 0$.

For $h$, the boundary conditions come from the limiting behaviour of $\phi$ at $\infty$. We also put a zero at the origin by construction, giving $h(\infty) = 1$, $h(0) = 0$.

**Theorem:** With the above ansatz, the basic vortex solution is given by the equations:

$$\frac{d^2h}{dr^2} + \frac{1}{r}\frac{dh}{dr} - \frac{1}{r^2}(1-f)^2 h + \frac{\lambda}{2}(1-h^2)h = 0,$$
$$\frac{d^2f}{dr^2} - \frac{1}{r}\frac{df}{dr} + (1-f)h^2 = 0.$$

*Proof:* In polar coordinates, the metric is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Hence remembering to use the inverse metric to raise indices, we have:

$$\frac{1}{4} f_{\mu\nu} f^{\mu\nu} = \frac{1}{2r^2}\left(\frac{\partial a_\theta}{\partial r}\right)^2.$$

Similarly, we find

$$\frac{1}{2}(D_\mu \phi)^* D^\mu \phi = \frac{1}{2}|\partial_r \phi|^2 + \frac{1}{2r^2}\left|\partial_\theta \phi - i a_\theta \phi\right|^2.$$

Hence substituting in the ansatz, we have $E =$

$$\int \left( \frac{1}{2r^2}(f')^2 + \frac{(h')^2}{2} + \frac{h^2}{2r^2}(1-f)^2 + \frac{\lambda(1-h^2)^2}{8} \right) r dr d\theta.$$

Computing the variational equations, we get the result. □

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These can be solved numerically, and give profiles like:

We can generalise this to $N$ vortices by setting $\phi = h(r)e^{iN\theta}$ instead. This gives a single, big vortex at the zero with multiplicity $N$ (i.e. they have all coalesced). They are unstable when $\lambda > 1$ (i.e. the regime where vortices repel).

## 2.5 Bogomolny vortices

In the critical case $\lambda = 1$, it's possible to derive Bogomolny equations for vortices (as opposed to direct method used above for basic vortex).

**Lemma:** We have

(i) The *covariant Leibniz rule*:

$$\partial_2(\phi^* D_1 \phi) = (D_2\phi)^* D_1 \phi + \phi^* D_2 D_1 \phi.$$

(ii) The *commutator*: $[D_1, D_2]\phi = -iB\phi = -if_{12}\phi$.

*Proof:* Both can be verified by direct calculation. □

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**Theorem:** For $\lambda = 1$, minimising the energy gives rise to the *Bogomolny equations*:

$$B - \frac{1}{2}(1 - \phi^*\phi) = 0,$$
$$D_1\phi + iD_2\phi = 0.$$

We also find the *Bogomolny bound* $E \geq \pi|N|$.

*Proof:* Perform a Bogomolny rearrangement. Focus on one sign that gives vortices, for clarity: $E =$

$$\frac{1}{2}\int_{\mathbb{R}^2}\left(\left(B - \frac{1}{2}(1-|\phi|^2)\right)^2 + |D_1\phi + iD_2\phi|^2\right) d^2x$$
$$+\frac{1}{2}\int_{\mathbb{R}^2}\left(B(1-\phi^*\phi) + i((D_2\phi)^*D_1\phi - (D_1\phi)^*D_2\phi)\right) d^2x.$$

We must compute the second integral. Using the covariant Leibniz rule (backwards) and the commutator we have that

$$B(1-|\phi|^2) + i((D_2\phi)^*D_1\phi - (D_1\phi)^*D_2\phi)$$
$$= B + i\partial_2(\phi^* D_1\phi) - i\partial_1(\phi^* D_2\phi).$$

Integrating the last two terms by parts, and using $D_1\phi, D_2\phi \to 0$ as $r \to \infty$, we see that they vanish. So we're left with the surface term

$$\frac{1}{2}\int_{\mathbb{R}^2} B \, d^2x = \pi N.$$

The Bogomolny equations follow from the first integral, and the Bogomolny bound follows from the leftover term (note we'd get $-N$ for antivortices, hence the modulus). □

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**Theorem:** If the Bogomolny equations are satisfied, the field equations are satisfied.

*Proof:* By differentiating the Bogomolny equations, whence constructing the field equations. □

## 2.6 Taube's equation and theorem

We can decouple the Bogomolny equations above by eliminating the gauge fields.

**Theorem:** If $\phi = e^{\frac{1}{2}u+i\chi}$, and the zeroes of the field $\phi$ are simple (all multiplicity $1$) and are at $\mathbf{X}_1$, $\mathbf{X}_2$, ... , $\mathbf{X}_N$, we have *Taube's equation*:

$$\nabla^2 u - e^u + 1 = \sum_{r=1}^{N} 4\pi\delta^2(\mathbf{x} - \mathbf{X}_r),$$

with boundary condition $u \to 0$ as $|\mathbf{x}| \to \infty$ from $|\phi| \to 1$.

*Proof:* Substituting into the second Bogomolny equation, we get

$$\partial_1(\frac{1}{2}u + i\chi) - ia_1 + i\partial_2(\frac{1}{2}u + i\chi) + a_2 = 0.$$

Compare real and imaginary parts and subsitute into the first equation, finding

$$B = \partial_1 a_2 - \partial_2 a_1 = -\frac{1}{2}\nabla^2 u,$$

which gives the result apart from the delta functions.

Where $\phi = 0$, $u \to -\infty$ logarithmically (as $u = \log|\phi|^2$). Hence need delta functions on RHS at zeroes of $\phi$. Near a zero, $u \approx 2\log|\mathbf{x} - \mathbf{X}|$, and quoting the Green's function of the Laplacian, we find

$$\nabla^2 u = 4\pi\delta^2(\mathbf{x} - \mathbf{X})$$

near one of the zeroes. The result follows. □

---

**Taube's Theorem:** Taube's equation has a unique solution for any choice of $\mathbf{X}_i$, $N$.

*Proof:* Not in course. □

---

## 2.7 Properties of Bogomolny vortices

**Theorem:** We have the following properties of Bogomolny vortices:

(i) the field $B$ is maximised at the vortex centres;

(ii) a single vortex has area $4\pi$;

(iii) $u$ is everywhere non-positive, implying $|\phi| \leq 1$ everywhere.

*Proof:* (i) $\phi = 0$ at vortex centres, then use Bogomolny.

(ii) One vortex has flux $2\pi$ from integral of $B$ giving $2\pi N$ for $N$ vortices. Now area $=$ flux$/B$, so at centre, get area $4\pi$.

(iii) By the maximum principle of the Laplacian. Where $u$ has a local maximum, $\nabla^2 u \leq 0$. Taube's equation then implies $e^u \leq 1$, and hence $u \leq 0$. □

A typical $u$ looks like:

---

What is the moduli space for vortices? They are completely described by $N$ zeros in $\mathbb{R}^2$ so naïvely looks like $\mathbb{R}^{2N}$. However...

**Theorem:** The $N$-vortex solution has moduli space $\mathcal{M}_N = \mathbb{R}^{2N}/S_N$.

*Proof:* Just notice that permuting the zeroes has no effect on the solution. □

On the other hand...

**Theorem:** $\mathbb{R}^{2N}/S_N \cong \mathbb{R}^{2N}$ as manifolds.

*Proof:* Let $Z_i = X_i^1 + X_i^2$ be the complex coordinates of $\mathbf{X}_i$. Defined the polynomial

$$P(z) = (z - Z_1)(z - Z_2)...(z - Z_N).$$

Choose instead to parametrise the moduli space by the coefficients of this polynomial. But they are symmetric functions of the roots, so quotient already accounted for. □

---

**Example:** Consider the two-vortex solution given by the complex polynomial

$$P(z) = (z - Z_1)(z - Z_2) = z^2 - (Z_1 + Z_2)z + Z_1 Z_2.$$

Here, $Z_1$ and $Z_2$ are good moduli.

In centre of mass coordinates, we can write $Z_1 = Z$ and $Z_2 = -Z$. Then $P(z) = z^2 - b$, where $b = Z^2$. The vortices are at $\pm Z = \pm\sqrt{b}$, and hence $b$ is a good modulus.

The fields evolve smoothly with $b$, but not with $Z$ as a result. Indeed, if $b$ hits zero, then there is a non-smooth result for $Z$. This effect can result in *vortex scattering*:

To obtain this, we have made $b$ dynamical. The vortices have kinetic energy $\frac{1}{2}g(b)\dot{b}^2$ for some function $g(b)$. This emphasises that the dynamics depend smoothly on $b$, but not on $Z$.

## 2.8 Bogomolny vortices on curved spaces

On a curved surface $\Sigma$, the energy of the field theory at critical coupling is:

$$E = \int\limits_{\Sigma} \left( \frac{1}{4} f_{ij} f_{kl} g^{ik} g^{jl} + \frac{1}{2} (D_i\phi)^* (D_j\phi) g^{ij} + \frac{1}{8}(1-\phi^*\phi)^2 \right) \sqrt{\det(g)} dy^1 dy^2,$$

where the metric is $g$. If there is a boundary of the surface $\Sigma$, we use the condition $|\phi| \to 1$ as we approach the boundary as before.

**Theorem:** On any surface, we can find *isothermal coordinates* $y^1$, $y^2$ in which the metric is

$$ds^2 = \Omega(y^1, y^2)((dy^1)^2 + (dy^2)^2),$$

*Proof:* Not required. □

In isothermal coordinates, the area element becomes $\Omega d^2y = \Omega dy^1 dy^2$.

---

Thus the energy becomes:

$$E = \frac{1}{2} \int\limits_{\Sigma} \left( \Omega^{-1} B^2 + |D_1\phi|^2 + |D_2\phi|^2 + \frac{\Omega}{4}(1-\phi^*\phi)^2 \right) dy^1 dy^2.$$

The Bogomolny rearrangement is identical to the planar one, giving:

**Theorem:** The energy is minimised when the Bogomolny equations are satisfied:

$$D_1\phi + iD_2\phi = 0,$$
$$B - \frac{\Omega}{2}(1-|\phi|^2) = 0.$$

We also have the Bogomolny bound $E \geq \pi|N|$, where $N$ is *defined* by

$$N = \frac{1}{2\pi} \int_\Sigma f = \frac{1}{2\pi} \int_\Sigma B d^2y.$$

$N$ is called the *first Chern number*. It no longer has the interpretation of a winding number, since we're on an abstract surface (e.g. could be on a sphere with no boundary!), however it is possible (and beyond the scope of the course) to show it is still an integer.

*Proof:* As for planar Bogomolny case. □

Taube's equation also has a natural generalisation to curved surfaces:

**Theorem:** If $|\phi|^2 = e^u$, we have *Taube's equation* when $E$ is minimised:

$$\nabla^2 u - \Omega e^u + \Omega = 4\pi \sum_{r=1}^{N} \delta^2(\mathbf{y} - \mathbf{Y}_r),$$

where the $\mathbf{Y}_r$ are the zeroes of $\phi$. This equation has a unique solution as in the planar case.

*Proof:* Not required. □

---

## 2.9 The Bradlow bound

**The Bradlow Bound:** Let $N$ be the number of vortices on a closed surface $\Sigma$. Then $N \leq A/4\pi$, where $A$ is the area of the surface.

*Proof:* Integrating the second Bogomolny equation directly, we have

$$2\int_\Sigma B\ d^2y + \int_\Sigma \Omega|\phi|^2\ d^2y = \int_\Sigma\ d^2y.$$

By the definition of $N$, we have $4\pi N \leq A$ immediately. □

This gives credence to the notion that *any* vortex takes up an area $4\pi$. It turns out it is possible to squish more vortices into a surface, but they turn out not to minimise energy, and are thus not Bogomolny.

If $4\pi N = A$, the vortices *dissolve*:

$$\int_\Sigma \Omega|\phi|^2\ d^2y = 0,$$

implying $\phi = 0$ everywhere, the vacuum.

---

## 2.10 Baptista's equation

**Definition:** The *Gaussian curvature* of a surface is

$$\kappa = -\frac{1}{2\Omega}\nabla^2\Omega.$$

**Definition:** The *vortex-modified metric* is defined by

$$\tilde{ds}^2 = \tilde{\Omega}((dy^1)^2 + (dy^2)^2),$$

where $\tilde{\Omega} = \Omega|\phi|^2 = \Omega e^u$.

**Theorem:** Let $\tilde{\kappa}$ be the Gaussian curvature on a surface with the vortex-modified metric. Then we have the *Baptista curvature equation*:

$$\left(\tilde{\kappa} + \frac{1}{2}\right)\tilde{ds}^2 = \left(\kappa + \frac{1}{2}\right) ds^2.$$

*Proof:* Just compute $\tilde{\kappa}$ directly using Taube's equation to evaluate $\nabla^2 u$. We find that

$$\left(\tilde{\kappa} + \frac{1}{2}\right) e^u = \kappa + \frac{1}{2}.$$

Multiplying both sides by $ds^2$, the standard metric, we get the result. □

Note that when $\kappa = -1/2$, we have that $\tilde{\kappa} = -1/2$ (since neither metric can vanish)!

## 2.11 Solutions in the hyperbolic plane

**Definition:** The *Poincaré disk* is a model of the *hyperbolic plane* in which the metric is

$$ds^2 = \frac{8}{(1-y_1^2-y_2^2)^2}\left((dy^1)^2+(dy^2)^2\right).$$

**Theorem:** The Poincaré disk has curvature $-1/2$.

*Proof:* Use $\nabla^2$ in polar coordinates, i.e.

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

and then this is easy to verify. □

---

It's often convenient to use a complex coordinate $z = y^1 + iy^2$ so that

$$ds^2 = \frac{8dzd\bar{z}}{(1-|z|^2)^2}$$

We can then consider how to get a possible $\Omega$.

Note that the vortex-modified metric is the metric of a surface with curvature $-1/2$ by Baptista's curvature equation, hence it is the metric of the hyperbolic plane.

Consider a conformal map $f : \mathbb{H}^2 \to \mathbb{H}^2$ (a map between two copies of the hyperbolic plane), from $z$-space, the initial space, to $w$-space, the vortex-modified space preserving the unit circle (the boundary).

Then the metric on $w$-space is

$$\tilde{ds}^2 = \frac{8}{(1-|w|^2)^2}dwd\bar{w} = \frac{8}{(1-|f(z)|)^2}\frac{df}{dz}\overline{\frac{df}{dz}}dzd\bar{z}.$$

Here, we have used the fact that

$$d\bar{w} = d(\overline{f(z)}) = \overline{\frac{df}{dz}}d\bar{z}.$$

Hence we identify

$$\tilde{\Omega} = \frac{8}{(1-|f(z)|^2)^2}\left|\frac{df}{dz}\right|^2, \quad \Omega = \frac{8}{(1-|z|^2)^2}.$$

By the definition of the vortex-modified metric then, we have

$$|\phi|^2 = \frac{\tilde{\Omega}}{\Omega} = \frac{(1-|z|^2)^2}{(1-|f(z)|^2)^2}\left|\frac{df}{dz}\right|^2.$$

Thus up to a choice of gauge we have:

**Theorem:** For any conformal map $f : \mathbb{H}^2 \to \mathbb{H}^2$ preserving $|z| = 1$, a vortex solution on the hyperbolic plane is given by

$$\phi = \frac{1-|z|^2}{1-|f(z)|^2}\frac{df}{dz}.$$

*Proof:* By the above work. □

**Example:** Consider $f(z) = z^n$. This gives:

$$\phi = \frac{n|z|^{n-1}}{1+|z|^2+...+|z|^{2n-2}},$$

the simplest multi-vortex solution in the hyperbolic plane. Note that this map preserves the unit circle, so preserves boundary conditions between the spaces.

How many vortices are there? Because we differentiated $z^n$ once, we ended up with a power of $z^{n-1}$. Note $z^{n-1}$ winds around $\infty$ with winding number $n-1$, and the only zeroes of $\phi$ are at $0$ (note generally the zeroes, i.e. the centres, of the vortices occur at $f'(z) = 0$).

Hence there are $n-1$ vortices at zero for this solution.

---

**Example:** The most general conformal function $f$ mapping $|z| = 1$ to $|w| = 1$ is given by

$$f(z) = \prod_{i=1}^{n}\left(\frac{z-c_i}{1-\overline{c_i}z}\right),$$

where $|c_i| < 1$ and $c_i$ is constant. This is called a *Blaschke product*.

---

**Example:** Consider $f(z) = z^2$. This gives the solution:

$$\phi = \frac{2|z|}{1+|z|^2} = \frac{2r}{1+r^2}$$

In polar coordinates. Note that as $r \to 1$ (the edge of the Poincaré disk), we get $\phi \to 1$, and so the boundary condition is satisfied.

---

## 2.12 Cone singularities

Notice that the metric $ds^2$ is smooth but $\tilde{ds}^2$ vanishes at the vortex centres. This implies that $\tilde{ds}^2$ has *conical singularities*.

The reason is as follows. A vortex of multiplicity $1$ has $\phi$ increasing linearly away from its centre (as above). Hence

$$\tilde{ds}^2 \sim r^2((dy^1)^2+(dy^2)^2) = r^2(dr^2+r^2d\theta^2).$$

Changing coordinates to $v = \frac{1}{2}r^2$ we find

$$\tilde{ds}^2 = dv^2 + 4v^2d\theta^2.$$

Finally changing coordinates to an angle $\beta \in [0, 4\pi]$, we get the standard flat metric $\tilde{ds}^2 = dv^2 + v^2d\beta^2$, albeit with a non-standard angle.

This non-standard angle is interpreted as the *cone angle*. For angles less than $2\pi$, this corresponds to cutting a wedge out of the plane and gluing the pieces back together:

Our angle cannot be interpreted in this way, but it is the same principle. It takes us two loops to go around the plane once.

---

## 2.13 Popov vortices

If we change the sign of $|D_1\phi|^2$ in the Abelian Higgs Lagrangian, we find that we get the Bogomolny equations:

$$\begin{aligned} D_1\phi + iD_2\phi &= 0, \\ B + \frac{1}{2}\Omega(1-|\phi|^2) &= 0. \end{aligned}$$

That is, the only thing that has changed is a sign in front of $\Omega$.

The Baptista curvature argument goes through as before, but this time we get:

$$\left(\tilde{\kappa} - \frac{1}{2}\right)\tilde{ds}^2 = \left(\kappa - \frac{1}{2}\right) ds^2$$

Thus we can construct explicit Bogomolny vortices for this new problem (similar to Abelian Higgs) on surfaces with curvature $1/2$ - this surface is a *sphere*.

---

ASIDE: STEREOGRAPHIC COORDINATES

Throughout the rest of the course, it will be very useful to convert between coordinates on the $2$-sphere and coordinates in the plane. We can do this by introducing *stereographic coordinates*.

The standard stereographic projection used in this course has the Riemann $2$-sphere centred on the of the plane, and of unit radius. We identify the top point, $(0,0,1)$ with $0$ and the bottom point, $(0,0,-1)$ with $-\infty$, as below:

We use a complex coordinate $z$ in the plane, and we use coordinates $(\theta,\phi)$ on the sphere. The coordinate $\phi$ is the angle in the plane, and the coordinate $\theta$ is the inclination between the vector $(0,0,-1)$ and our position on the sphere.

**Theorem:** The complex coordinate $z$ corresponding to coordinates $(\theta,\phi)$ is $z = \tan(\frac{1}{2}\theta)e^{i\phi}$.

*Proof:* Clearly the argument of our complex number $z$ is $\phi$, so write $z = |z|e^{i\phi}$. Draw a diagram to get the modulus:

Use isosceles triangles and sine rule to finish. □

**Theorem:** The sphere coordinate $(\theta,\phi)$ corresponding to the complex plane coordinate $z$ is given by

$$(\theta,\phi) = \left(\arccos\left(\frac{1-|z|^2}{1+|z|^2}\right), \arccos\left(\frac{z+\overline{z}}{2|z|}\right)\right)$$

*Proof:* Write $z = \tan(\frac{1}{2}\theta)e^{i\phi}$ as above. Then

$$|z|^2 = \frac{\sin^2(\frac{1}{2}\theta)}{\cos^2(\frac{1}{2}\theta)} = \frac{1-\cos(\theta)}{1+\cos(\theta)}.$$

Rearrange and invert to get $\theta$. Similarly, notice that

$$\frac{z}{\overline{z}} = e^{2i\phi} \Rightarrow \cos(2\phi) = \frac{1}{2}\left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) = \frac{1}{2}\left(\frac{z^2+\overline{z}^2}{|z|^2}\right).$$

Now use $\cos(2\phi) = 2\cos^2(\phi) - 1$ to get result. □

---

It is also useful to be able to convert between Cartesian coordinates for the sphere and their corresponding complex coordinates.

**Theorem:** The complex $z$ coordinate corresponding to the point $(x_1,x_2,x_3)$ on the $2$-sphere is:

$$z = \frac{x_1 + ix_2}{1+x_3}.$$

*Proof:* Note that $\phi = \arctan(x_2/x_1)$ and $\theta = \arccos(x_3)$. So substituting in $z = \tan(\frac{1}{2}\theta)e^{i\phi}$, we get the result. Best to use

$$\tan^2\left(\frac{1}{2}\theta\right) = \frac{1-\cos(\theta)}{1+\cos(\theta)} = \frac{1-x_3}{1+x_3}.$$

and also $x_1^2 + x_2^2 + x_3^2 = 1$. □

**Theorem:** The sphere coordinates $(x_1,x_2,x_3)$ corresponding to the complex coordinate $z$ are given by

$$(x_1,x_2,x_3) = \frac{1}{1+|z|^2}\left(z+\overline{z}, -i(z-\overline{z}), 1-|z|^2\right)$$

*Proof:* Use $(x_1,x_2,x_3) = (\sin\theta\cos\phi, \sin\phi\sin\theta, \cos\theta)$ and result above giving $(\theta,\phi)$ in terms of $z$. □

We are now ready to do geometry on the sphere. Using the above calculations to substitute into

$$ds^2 = 2(d\theta^2 + \sin^2(\theta)d\phi^2),$$

($2$ is here to ensure curvature $1/2$) we find that the sphere metric is

$$ds^2 = \frac{8dzd\bar{z}}{(1+|z|^2)^2}$$

in terms of the stereographic coordinate $z = \tan(\frac{1}{2}\theta)e^{i\phi}$. This gives, analogously to the Poincaré disk case, the vortex solution

$$\phi = \frac{1+|z|^2}{1+|f(z)|^2}\frac{df}{dz}$$

for any conformal $f$ (in fact, a conformal map from a sphere to a sphere is holomorphic). This time, the acceptable $f$'s turn out to be *rational maps* (rather than Blaschke products).

We'll discuss these later in the course, but you already know what they are: maps of the form $R(z) = p(z)/q(z)$ for $p$, $q$ polynomials. The *Wronskian* of the map is defined by $W(z) = p'(z)q(z) - p(z)q'(z)$.

---

The centres of the vortices occur when $f'(z)$, which is precisely when the Wronksian is zero. In the generic case that $p$ and $q$ have equal degrees, say $n$, we get that $W(z)$ is of degree $2n-1$.

Or do we? We'll see in a second that the Wronskian is generically of degree $2n-2$, because the first terms cancel in $p'q - q'p$. Therefore there are $N = 2n-2$ vortices in the generic case.

**Definition:** $N$ is called the *Popov vortex number*.

Surprisingly, it's always even - even though the Bradlow bound seemingly permits all values of $N$ less than $A/4\pi$.

---

# 3 Degrees and rational maps

## 3.1 The degree of a mapping

We generalise winding number to maps $f : M \to N$ where $M$, $N$ are closed (i.e. no boundary), connected, oriented manifolds of equal dimension $d$.

**Definition:** Let $\omega$ be a normalised volume form on $N$, i.e. a $d$-form on $N$ satisfying

$$\int_N \omega = 1.$$

The *degree* of such a map is

$$\deg(f) = \int_M f^*\omega,$$

where $f^*\omega$ is the *pull-back* of $\omega$ to $M$.

---

**Example:** Let $f : \mathbb{R}^2 \to S^2$ (can compactify $\mathbb{R}^2$ to make it closed), given by $(\theta, \phi) = (\theta(x,y), \phi(x,y))$. Choose the standard volume form on $S^2$ normalised to $1$:

$$\omega = \frac{1}{4\pi}\sin(\theta)d\theta \wedge d\phi.$$

The pull-back is then:

$$\begin{aligned} f^*\omega &= \frac{1}{4\pi}\sin(\theta(x,y))\left(\frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy\right) \wedge \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy\right) \\ &= \frac{1}{4\pi}\sin(\theta(x,y))\left(\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial\theta}{\partial y}\frac{\partial\phi}{\partial x}\right)dx \wedge dy. \end{aligned}$$

This can be integrated over $\mathbb{R}^2$ to get the degree of $f$.

---

## 3.2 Why is degree an integer?

**Theorem:** The choice of volume form in degree of a map is arbitrary.

*Proof:* Let $\omega$, $\tilde{\omega}$ be normalised volume forms on $N$. Then

$$\int_N (\omega - \tilde{\omega}) = 0.$$

This implies, by results in homology theory, that $\omega - \tilde{\omega}$ is closed, i.e. $\omega - \tilde{\omega} = d\alpha$ for some $\alpha$, a $d-1$ form.

Under pull-back we have (because this is just a change of coordinates):

$$f^*(\omega - \tilde{\omega}) = f^*(d\alpha) = d(f^*\alpha).$$

Then

$$\int_M f^*(\omega - \tilde{\omega}) = 0 \;\Rightarrow\; \int_M f^*\omega = \int_M f^*\tilde{\omega}. \quad \square$$

---

**Theorem:** $\deg(f)$ is the number of preimages of a generic point $p \in N$, counted according to orientation.

*Proof:* Since $\omega$ is arbitrary, choose it to be highly localised around some $p \in N$. Then $\omega$ is zero outside of $\Sigma$, some small neighbourhood of $p$.

Let $f_1^{-1}(p)$, $f_2^{-1}(p)$, ... be the preimage points of $p$ in $M$. Let $f_1^{-1}(\Sigma)$, $f_2^{-1}(\Sigma)$, ... be the preimage regions. Then by the change of coordinates formula:

$$\int_{f_i^{-1}(\Sigma)} f^*\omega = \pm\int_\Sigma \omega = \pm\int_N \omega = \pm 1.$$

We get the $\pm$ sign dependent on the sign of the Jacobian of $f$ in a neighbourhood of $f_i^{-1}(\Sigma)$. It's $+$ for positive orientation, $-$ for negative orientation. 'Generic point' in the Theorem translates to $f$ not having zero Jacobian.

Therefore:

$$\deg(f) = \int_M f^*\omega = \sum_i \int_{f_i^{-1}(\Sigma)} f^*\omega = \sum_{\substack{\text{preimages} \\ \text{of } p}} (\pm 1). \quad \square$$

---

We have the immediate Corollaries:

- The degree is an integer.
- Since the degree is an integer, it is invariant under continuous changes of $\omega$, $p$ and $f$. So degree is a topological invariant.

---

**Example:** Consider a map $\phi : S^1 \to S^1$ with degree $1$. A typical $\phi$ looks like:

This map clearly has winding number $1$. At any generic point, we have $1$ preimage, accounting for orientation, as in the diagram (unless the point has zero derivative!). So degree is a generalisation of winding number.

---

## 3.3 Rational maps

**Definition:** A *rational map* $R$ is a map $R : S^2 \to S^2$ (where we view each $S^2$ as a copy of the Riemann sphere) such that

$$R(z) = \frac{p(z)}{q(z)},$$

where $p$, $q$ are polynomials. We assume $p$, $q$ have no common roots.

---

**Theorem:** A rational map $R(z)$ is holomorphic and orientation preserving.

*Proof:* Just a function of $z$ so holomorphic.

To see the map is orientation preserving, consider the area element on the sphere, $r d\theta \wedge d\phi$. In terms of the stereographic coordinate $z = \tan(\frac{1}{2}\theta)e^{i\phi}$, we can rewrite this as

$$dA = 2i \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

Therefore on the target sphere the area element is

$$\tilde{dA} = 2i \frac{dz \wedge d\bar{z}}{(1 + |R(z)|^2)^2} \left| \frac{dR}{dz} \right|^2$$

But $|R'(z)|^2 \geq 0$, so this has the same sign as $dA$. So orientation is preserved. $\square$

---

## 3.4 Algebraic degree

**Definition:** The *algebraic degree* of a rational map $R$ is

$$\deg_{\text{alg}}(R) = \max\{\deg(p), \deg(q)\},$$

where $R(z) = p(z)/q(z)$, $p(z)$, $q(z)$ coprime polynomials.

**Examples:** (i) $z^2$ is degree $2$; (ii) $1/z^2$ is degree $2$; (iii) $z/(z^2 - 1)$ is degree $2$.

---

**Theorem:** The *topological degree*, i.e. $\deg(R)$, i.e. the degree of the mapping, is equal to the algebraic degree.

*Proof:* Let $c$ be a generic point in the target space, so that preimage $R$ satisfies $R(z) = c$. Then

$$p(z) - cq(z) = 0.$$

For a generic point, the fundamental theorem of algebra gives $\max\{\deg(p), \deg(q)\} = \deg_{\text{alg}}(R)$ distinct roots.

Since rational maps are orientation preserving the preimages all have positive orientation. Hence the number of preimages is $\deg(R) = \deg_{\text{alg}}(R)$. $\square$

---

## 3.5 The Wronskian

**Definition:** The *Wronskian* $W(z)$ of a rational map $R(z) = p(z)/q(z)$ is defined by

$$W(z) = p'(z)q(z) - p(z)q'(z).$$

The *Wronskian points* are the zeroes of $W$.

---

**Theorem:** Under a Möbius transformation is performed on $R(z)$, then the Wronskian points are unchanged.

*Proof:* The new rational function is:

$$\tilde{R}(z) = \frac{\alpha R(z) + \beta}{\gamma R(z) + \delta} = \frac{\alpha p + \beta q}{\gamma p + \delta q}.$$

The new Wronskian is:

$$(\alpha p' + \beta q')(\gamma p + \delta q) - (\gamma p' + \delta q')(\alpha p + \beta q) = (\alpha\delta - \beta\gamma)W(z),$$

and for a Möbius transformation, $\alpha\delta - \beta\gamma \neq 0$. $\square$

**Theorem (Properties of Wronskian):**

(i) $R'(z) = 0$ iff $W(z) = 0$.

(ii) For generic polynomials $p$, $q$, we have
$$\deg(W) = 2\deg(R) - 2.$$

(iii) If $R(z)$ has a symmetry, then $W(z)$ also has the same symmetry.

(iv) If $W$ has degree less than $2\deg(R) - 2$, then the difference is made up by zeroes of $W$ lying at $\infty$.

*Proof:* (i) Trivial. (ii) Substitute general polynomials $p$, $q$ in to check.

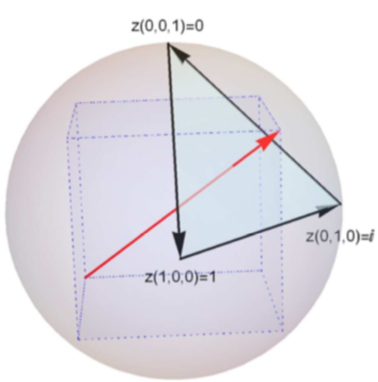
(iii), (iv) Not required. □

---

## 3.6 Example: tetrahedral symmetry

**Example:** Consider the rational map
$$R(z) = \frac{\sqrt{3}iz^2 - 1}{z(z^2 - \sqrt{3}i)}$$
whose Wronskian is
$$\begin{aligned} W(z) &= 2\sqrt{3}iz(z^3 - \sqrt{3}iz) - (3z^2 - \sqrt{3}i)(\sqrt{3}iz^2 - 1) \\ &= -\sqrt{3}iz^4 + 6z^2 - \sqrt{3}i \end{aligned}$$
This has four roots, which are the Wronskian points. We claim that in Cartesian coordinates they are given by:
$$\frac{1}{\sqrt{3}}(1,1,1),\ \frac{1}{\sqrt{3}}(-1,-1,1),\ \frac{1}{\sqrt{3}}(-1,1,-1),\ \frac{1}{\sqrt{3}}(1,-1,-1).$$
The corresponding complex points are:
$$z = \frac{1+i}{1+\sqrt{3}},\ \frac{-1-i}{1+\sqrt{3}},\ \frac{-1+i}{\sqrt{3}-1},\ \frac{1-i}{\sqrt{3}-1}.$$
Multiplying the factors $(z - z_i)$ together, we find the polynomial above, as expected.

When plotted on the Riemann sphere, these roots, together with a corresponding set of roots from another tetrahedron, form the vertices of a cube. The normals to the midpoints of the faces of the cube meet the Riemann sphere at $\infty, 0, \pm 1, \pm i$, which is convenient:




The Wronskian points are tetrahedrally symmetric in the following sense:

- Consider rotation $120°$ about an axis through one of the vertices, say parallel to $(1,1,1)$. From the cube interpretation, this is equivalent to rotating vertices of the cube. Indeed, this is equivalent to sending $0 \mapsto 1 \mapsto i \mapsto 0$ on the Riemann sphere.

  The $120°$ rotation of the tetrahedron is therefore represented by the Möbius map:
  $$k(z) = \frac{iz+1}{-iz+1}.$$
  Under this rotation, we have
  $$R(k(z)) = \frac{(1 - i\sqrt{3})R(z) + (\sqrt{3} - i)}{-(\sqrt{3} + i)R(z) + (1 + i\sqrt{3})},$$
  and hence this is a symmetry, as defined later in the course.

- We can also consider a *reflection* along one of the tetrahedron's axes of symmetry. We can check the diagram that
  $$k(z) = iz$$
  is such a reflection. Then
  $$R(k(z)) = \frac{-\sqrt{3}iz^2 - 1}{iz(-z^2 - \sqrt{3}i)} = i\overline{R(z)}.$$
  Hence this is also a symmetry.

# 4 Skyrmions

## 4.1 The Skyrme Lagrangian and SSB

**Definition:** The *Skyrme field* is a map $U : \mathbb{R}^{3+1} \to S^3$. We view $U$ as $SU(2)$-valued, since $SU(2) \cong S^3$ as manifolds. We write

$$U(\mathbf{x}, t) = \sigma(\mathbf{x}, t)I_2 + i\boldsymbol{\pi}(\mathbf{x}, t) \cdot \boldsymbol{\tau},$$

where $I_2$ is the identity matrix and $\boldsymbol{\tau}$ are the Pauli matrices. $\sigma$ is called the *sigma field* and $\boldsymbol{\pi}$ are called the *pion fields*. We have $\sigma^2 + |\boldsymbol{\pi}|^2 = 1$ for $U$ to be in $SU(2)$.

**Definition:** The *Skyrme Lagrangian* is

$$L = \int d^3\mathbf{x} \bigg( -\frac{F_\pi^2}{16} \mathrm{Tr}\left( (\partial_\mu U)U^{-1}(\partial^\mu U)U^{-1} \right) + \frac{1}{32e^2} \mathrm{Tr}\left( [(\partial_\mu U)U^{-1}, (\partial_\nu U)U^{-1}][(\partial^\mu U)U^{-1}, (\partial^\nu U)U^{-1}] \right) \bigg).$$

---

**Definition:** A *chiral transformation* is a transformation of the form $U \mapsto O_1 U O_2$ where $O_1, O_2 \in SU(2)$.

Clearly the Skyrme Lagrangian is invariant under chiral transformations. This means:

**Theorem:** The symmetry group of the Skyrme Lagrangian is

$$\frac{SU(2) \times SU(2)}{\mathbb{Z}_2}.$$

*Proof:* Via the standard proof in Standard Model course. □

---

Note that the theory undergoes spontaneous symmetry breaking. The vacuum can be chosen WLOG to be $U = I_2$, i.e. $\sigma = 1$ and $\boldsymbol{\pi} = \mathbf{0}$. Performing perturbations around the vacuum, we get $\boldsymbol{\pi}$ particles but no $\sigma$ particle.

**Theorem:** The unbroken symmetry group is $SU(2)/\mathbb{Z}_2 \cong SO(3)$.

*Proof:* We need to preserve the vacuum. So $O_1 I O_2 = I$, and it follows that $O_1 = O_2^{-1}$. Result follows. □

**Definition:** The symmetry group $SO(3)$ is called the *isospin symmetry* of the Skyrme Lagrangian.

---

## 4.2 Notation and field equations

To ease notation, write $R_\mu = \partial_\mu U U^{-1}$. As we saw in Symmetries, Fields and Particles, $R_\mu \in \mathcal{L}(SU(2))$, the Lie algebra of $SU(2)$.

Also rescale space and time to remove the $F_\pi$ and $e$ constants in the Skyrme Lagrangian. Then the Skyrme Lagrangian becomes:

$$L = \int \left( -\frac{1}{2}\mathrm{Tr}(R_\mu R^\mu) + \frac{1}{16}\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right) d^3\mathbf{x}.$$

**Theorem:** The field equations of the the Skyrme Lagrangian are:

$$\partial_\mu \left( R^\mu + \frac{1}{4}[R_\nu, [R^\nu, R^\mu]] \right) = 0.$$

The linearised equations are $\partial_\mu \partial^\mu \boldsymbol{\pi} = 0$.

*Proof:* This is a tedious but straightforward calculation. It helps to note a variation of $U$ has the form $\delta U = \epsilon X U$, with $X$ in the Lie algebra of $SU(2)$. □

---

## 4.3 Derrick's Theorem for Skyrmions

As usual, we consider static field configurations. Then the energy takes the form:

$$E = \int \left( -\frac{1}{2}\mathrm{Tr}(R_i R_i) - \frac{1}{16}\mathrm{Tr}([R_i, R_j][R_i, R_j]) \right) d^3\mathbf{x}.$$

Write:

$$E_2 = \int -\frac{1}{2}\mathrm{Tr}(R_i R_i) d^3\mathbf{x},\ E_4 = \int -\frac{1}{16}\mathrm{Tr}([R_i, R_j][R_i, R_j]) d^3\mathbf{x}$$

Note that $E_2, E_3 \geq 0$ because $\mathcal{L}(SU(2))$ is of compact type, so the Killing form is negative definite.

**Derrick's Theorem:** $E_2 = E_4$ for any soliton solution.

*Proof:* Let $U(\mathbf{x})$ be a soliton solution, i.e. a stationary point of energy satisfying the field equations. Then $\tilde{U}(\mathbf{x}) = U(\lambda \mathbf{x})$ will minimise the energy when $\lambda = 1$. The proof then proceeds exactly as in the proof of Derrick's Theorem for kinks. □

---

## 4.4 Strain eigenvalues and energy

A more convenient representation of the Skyrme energy is through *strain eigenvalues*.

**Definition:** The *strain tensor* is $D_{ij} = -\frac{1}{2}\mathrm{Tr}(R_i R_j)$. The eigenvalues of the strain tensor are denoted $\lambda_1^2$, $\lambda_2^2$ and $\lambda_3^2$. We call the $\lambda_i$ the *principal strains*.

The strain eigenvalues represent how much stretching our Skyrme map $U$ does. Consider the diagram:

A ball of radius $\epsilon \ll 1$ is mapped to some ellipsoid on $S^3_1 \cong SU(2)$, the unit 3-sphere. Locally then, we can view the map as a linear stretching. The ellipsoid has principal axes of lengths $\lambda_1\epsilon$, $\lambda_2\epsilon$ and $\lambda_3\epsilon$, where the $\lambda_i$ are the strains as above.

**Theorem:** In terms of the strain eigenvalues, the Skyrme energy is:

$$E = \int\limits_{\mathbb{R}^2} \left( (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + (\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2) \right) d^3\mathbf{x}.$$

*Proof:* To begin with, it is best to rewrite $D_{ij}$ in a different form. Near $I$, we may write

$$U(\mathbf{x}) = I + i\boldsymbol{\pi} \cdot \boldsymbol{\tau},$$

from which it follows $R_i = i(\partial_i \boldsymbol{\pi}) \cdot \boldsymbol{\tau}$. Then

$$D_{ij} = -\frac{1}{2}\mathrm{Tr}(R_iR_j) = \frac{1}{2}\partial_i\pi^a\partial_j\pi^b\mathrm{Tr}(\tau^a\tau^b) = \partial_i\boldsymbol{\pi} \cdot \partial_j\boldsymbol{\pi}.$$

Now let's begin rewriting the energy in terms of the strains. The first term in the Lagrangian is just the trace of the strain tensor, hence

$$D_{ii} = -\frac{1}{2}\mathrm{Tr}(R_iR_i) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

by definition.

To get the remaining terms, we use the alternative form we derived above. We first note that

$$[R_i, R_j] = -\partial_i\pi^a\partial_j\pi^b[\tau^a, \tau^b] = -2i\epsilon^{abc}\partial_i\pi^a\partial_j\pi^b\tau^c.$$

Hence $\mathrm{Tr}([R_i, R_j][R_i, R_j]) =$

$$\begin{aligned} &-4\epsilon^{abc}\epsilon^{def}\partial_i\pi^a\partial_j\pi^b\partial_i\pi^d\partial_j\pi^e\mathrm{Tr}(\tau^c\tau^f) \\ &= -8\epsilon^{abc}\epsilon^{dec}\partial_i\pi^a\partial_j\pi^b\partial_i\pi^d\partial_j\pi^e \\ &= -8(\partial_i\boldsymbol{\pi} \cdot \partial_i\boldsymbol{\pi}\partial_j\boldsymbol{\pi} \cdot \partial_j\boldsymbol{\pi} - \partial_i\boldsymbol{\pi} \cdot \partial_j\boldsymbol{\pi}\partial_i\boldsymbol{\pi} \cdot \partial_j\boldsymbol{\pi}) \\ &= -8(D_{ii}D_{jj} - D_{ij}D_{ji}). \end{aligned}$$

Now use the following result: $\mathrm{Tr}(A^2) = \sum_i \mu_i^2$ where $\mu_i$ are the eigenvalues of $A$. Then we have

$$\begin{aligned} D_{ii}D_{jj} - D_{ij}D_{ji} &= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^2 - (\lambda_1^4 + \lambda_2^4 + \lambda_3^4) \\ &= 2\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \end{aligned}$$

whence the result follows. □

---

**Lemma:** The volume of the unit $d$-sphere is:

$$\mathrm{Vol}(S_1^d) = \frac{\pi^{d/2}}{\Gamma(d/2)}$$

*Proof:* Note that

$$(\sqrt{\pi})^d = \int_{\mathbb{R}^d} \prod_{i=1}^{d} (e^{-x_i^2} dx_i) = \mathrm{Vol}(S^{d-1}) \int\limits_0^\infty e^{-r^2} r^{d-1} dr.$$

Let $u = r^2$ and the use the definition of the gamma function. □

**Theorem:** The degree of the mapping $U : \mathbb{R}^3 \to S_1^3$ is given by

$$B = \frac{1}{2\pi^2} \int\limits_{\mathbb{R}^3} \lambda_1\lambda_2\lambda_3 \; d^3\mathbf{x}.$$

*Proof:* Note $\mathbb{R}^3$ is not compact, but $U \to I_2$ as $|\mathbf{x}| \to \infty$ allowing us to compactify and identify $\mathbb{R}^3 \cong S^3_\infty$. Note that $\lambda_1\lambda_2\lambda_3$ is the factor by which volumes change under $U$; therefore the (non-normalised) standard volume form on $S_1^3$ has pull-back:

$$\lambda_1\lambda_2\lambda_3 d^3\mathbf{x}.$$

It follows that

$$\int \lambda_1\lambda_2\lambda_3 \; d^3\mathbf{x} = \mathrm{Vol}(S_1^3)B = 2\pi^2 B,$$

where the volume of $S_1^3$ is included to guarantee normalisation. □

**Theorem (Fadeev-Bogomolny Bound):** We have $E \geq 12\pi^2|B|$.

*Proof:* We complete the square in the energy to get:

$$E = \int\limits_{\mathbb{R}^3} \left( (\lambda_1 - \lambda_2\lambda_3)^2 + (\lambda_2 - \lambda_3\lambda_1)^2 + (\lambda_3 - \lambda_1\lambda_2)^2 \right) d^3\mathbf{x} + 12B\pi^2.$$

Note there are two options for sign when completing the square, which gives the modulus. The result follows immediately. □

---

Note that it is *impossible* to attain this bound. Equality occurs iff $\lambda_1 = \lambda_2\lambda_3$, $\lambda_2 = \lambda_3\lambda_1$ and $\lambda_3 = \lambda_1\lambda_2$ hold everywhere; it is possible to show the only solutions are:

- $\lambda_1 = \lambda_2 = \lambda_3 = 0$, i.e. $U$ is constant. This is the vacuum configuration $U = I$ everywhere.
- $\lambda_1 = \lambda_2 = \lambda_3 = 1$. then there is *no strain* and the map preserves lengths locally. This is impossible since we have to scrunch up $\mathbb{R}^3$ in some places to get to $S_1^3$, since the two spaces are not isometric.

### 4.5 The hedgehog Skyrmion

The $B = 1$ Skyrmion solution comes from the ansatz:

$$U(\mathbf{x}) = \cos(f(r))I + i\sin(f(r))\hat{\mathbf{x}} \cdot \boldsymbol{\tau}.$$

Let's examine the form of this map:

- Note that any $2$-sphere of fixed radius $r$ in $\mathbb{R}^3$ gets mapped to a $2$-sphere in $SU(2)$ space of radius $\sin(f(r))$, since we can view $\cos(f(r))$ fixed, leaving the image only dependent on $\sin(f(r))\hat{\mathbf{x}}$.

  Hence we have *split up the angular and radial dependencies*.

- The boundary conditions require $U \to I$ as $|\mathbf{x}| \to \infty$ and $U \to -I$ as $|\mathbf{x}| \to 0$ (the map must cover the target space, and thus $-I$ must have a preimage, WLOG put it at the origin). Therefore $f(0) = \pi$ and $f(\infty) = 0$.

Because of our interpretation of a fixed $2$-sphere of radius $r$ mapping to a fixed $2$-sphere of radius $\sin(f(r))$, we can compute the strains.

**Theorem:** The radial strain is $\lambda_1 = -f'$ and the angular strains are

$$\lambda_2 = \lambda_3 = \frac{\sin(f(r))}{r}.$$

*Proof:* Consider a sphere of radius $r \ll 1$ centred at $\mathbf{0}$ in the domain. This is mapped to a sphere of radius $\sin(f(r))$. The metric on the original $2$-sphere is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2,$$

and the metric on the new $2$-sphere is

$$\tilde{ds}^2 = \sin^2(f(r))d\theta^2 + \sin^2(f(r))\sin^2(\theta)d\phi^2.$$

Comparing the two, we see that the angular strains are indeed:

$$\lambda_2 = \lambda_3 = \frac{\sin(f(r))}{r}.$$

Now consider the radial metrics. For small $r$, we have that the sphere in the domain is mapped to a sphere of radius $\sin(f(r))$ in the target. Hence

$$\sin(f(r)) \approx f'(r)r.$$

near $r = 0$. Since $f$ is decreasing we have $f' < 0$, so the strain is $\lambda_1 = -f'$. □

**Theorem:** The hedgehog Skyrmion indeed has degree $B = 1$.

*Proof:* Simply substitute strains into the formula for $B$ above and integrate. □

**Theorem:** The energy of the hedgehog Skyrmion is given by:

$$E = \int_0^\infty \left( (f')^2 + \frac{2\sin^2(f)}{r^2}(1 + (f')^2) + \frac{\sin^4(f)}{r^4} \right) 4\pi r^2 \, dr.$$

*Proof:* Again, substitute into general formula. □

---

We can now analyse the field equations for $f$. We find:

**Theorem:** $f$ obeys the ODE: $0 =$

$$(-r^2 - 2\sin^2(f))f'' - 2rf' + \sin(2f)(1 - (f')^2) + \frac{2\sin^3(f)\cos(f)}{r^2}.$$

*Proof:* Simply compute EL equations of above integral. □

Thus linearising this equation, i.e. expanding $\sin(f) \approx f$, $\cos(f) \approx 1$, etc, we see that $f$ obeys:

$$-r^2 f'' - 2rf' + 2f = 0$$

as $r \to \infty$. Solving with the ansatz $f = r^k$, we find $k = 1$, $k = -2$. The $k = 1$ solution is not useful as it is unbounded.

Therefore the asymptotics of $f$ as $r \to \infty$ are given by

$$f \sim \frac{C}{r^2},$$

and hence the asymptotics of the Skyrme field are:

$$U(\mathbf{x}) = \left(1 - \frac{C^2}{r^4}\right) I + i\frac{C}{r^2}\hat{\mathbf{x}} \cdot \boldsymbol{\tau}.$$

## 4.6 The rational map approximation

Generalise the hedgehog. Try an ansatz of the form

$$U(\mathbf{x}) = \cos(f(r))I + i\sin(f(r))\hat{\mathbf{n}}_{R(z)}\cdot\boldsymbol{\tau},$$

where $\hat{\mathbf{n}}$ is the point on the Riemann sphere realised as $S^2$ corresponding to the point $R(z)$ in the complex plane. Also, $z = \tan(\frac{1}{2}\theta)e^{i\phi}$, where $(\theta,\phi)$ are the angular coordinates of $\mathbf{x}$, in the domain.

**Theorem:** We have

$$\hat{\mathbf{n}}_{R(z)} = \frac{1}{1+|R|^2}(R+\overline{R}, -i(R-\overline{R}), 1-|R|^2)^T.$$

*Proof:* Just use the result from the above section on stereographic coordinates (under Popov vortices). □

---

**Theorem:** If $R$ has degree $n$, then the degree of the Skyrme field is $B = n$.

*Proof:* Count preimages. $f$ is a monotonic function, so the only times we can have multiple preimages are if $\hat{\mathbf{n}}_{R(z)}$ maps to the same points on $S^2$ for different values of $z$.

But then the number of preimages is the same as the number of preimages of $R(z)$. It follows that the degrees are equal. □

---

**Theorem:** The strain eigenvalues are

$$\lambda_1 = -f', \quad \lambda_2 = \lambda_3 = \frac{\sin(f(r))}{r}\left(\frac{1+|z|^2}{1+|R(z)|^2}\right)\left|\frac{dR}{dz}\right|.$$

*Proof:* The radial argument is as before. The angular argument comes from that fact that the metric on $S^2_r$ in the domain is:

$$ds^2 = \frac{r^2 dz d\overline{z}}{(1+|z|^2)^2}.$$

The metric on $S^2_{\sin(f(r))}$ is

$$\tilde{ds}^2 = \frac{\sin^2(f(r))dRd\overline{R}}{(1+|R|^2)^2} = \left|\frac{dR}{dz}\right|^2\frac{\sin^2(f(r))dzd\overline{z}}{(1+|R|^2)^2}.$$

Hence the angular strains have product:

$$\lambda_2\lambda_3 = \frac{\sin^2(f(r))}{r^2}\left(\frac{1+|z|^2}{1+|R|^2}\right)^2\left|\frac{dR}{dz}\right|^2.$$

By the symmetry, $\lambda_2 = \lambda_3$, and hence they are equal to the square root of the above. □

**Theorem:** The energy is given by

$$E = \int_0^\infty \left((f')^2 + 2B\frac{\sin^2(f)}{r^2}(1+(f')^2) + \mathcal{I}\frac{\sin^4(f)}{r^4}\right)4\pi r^2 dr.$$

where $\mathcal{I}$ is a completely angular integral.
*Proof:* Use the standard form in terms of the strain eigenvalues. Need to use:

$$B = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\lambda_1\lambda_2\lambda_3\, d^3\mathbf{x}$$

$$= \underbrace{-\frac{1}{2\pi^2}\int_0^\infty f'\sin^2(f(r))\,dr}_{=1}\int_{\mathbb{R}^2}|R'|^2\left(\frac{1+|R|^2}{1+|z|^2}\right)d\Omega.$$

to replace an expression in the second term. □

---

To find the approximate rational map Skyrmions, we minimise $E$. We first minimise $\mathcal{I}$ numerically via a choice of $R(z)$. Remarkably, $R(z)$ often has a lot of symmetry:

- For $B = 1$, $R_1(z) = z$. This gives rise to $\hat{\mathbf{n}}_{R(z)} = \hat{\mathbf{x}}$. This is spherically symmetric.
- For $B = 2$, $R_2(z) = z^2$. This is toroidally symmetric.
- For $B = 3$, $R_3(z) = \dfrac{\sqrt{3}iz^2 - 1}{z^3 - iz\sqrt{3}}$. This is tetrahedrally symmetric.

---

## 4.7 Comparison to true Skyrmions

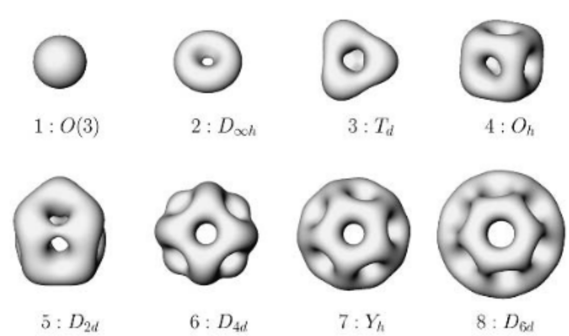
**Theorem:** For $B < 8$, the approximate rational map Skyrmions have the *same symmetries* as the true Skyrmions.

---

**Definition:** The *baryon density* is $\mathcal{B} = \lambda_1\lambda_2\lambda_3$.

By the above, $\mathcal{B} \propto R'(z)$, and hence the baryon number density *vanishes at the Wronskian points*. It follows that the approximate Skyrmions have holes at $2B-2$ points.

It turns out the true Skyrmions also have the same number of holes:



## 4.8 Pion mass terms

In the original Skyrme Lagrangian, we may also include the pion mass terms:

$$\int -\frac{1}{8}F_\pi^2 m_\pi^2 \operatorname{Tr}(I_2 - U)\, d^3\mathbf{x}.$$

**Theorem:** For $|\boldsymbol{\pi}| \ll 1$, this is manifestly as mass term.

*Proof:* We have

$$\operatorname{Tr}(I_2 - U) = \operatorname{Tr}(I_2 - \sigma I_2) = 2(1-\sigma).$$

since the Pauli matrices are traceless. Now

$$1 - \sigma = 1 - \sqrt{1 - |\boldsymbol{\pi}|^2} \sim \frac{1}{2}\boldsymbol{\pi}^2.$$

Therefore the mass term becomes:

$$\int -\frac{1}{8}F_\pi^2 m_\pi^2 \boldsymbol{\pi}\cdot\boldsymbol{\pi}\, d^3\mathbf{x}.$$

The normal rescaling of space normalises this term to a mass term. □

The potential energy above is very high for $U = -I_2$. These are regions where $\cos(f) = -1$, i.e. $f = \pi$, thus $\sin(f) = 0$, so the baryon density $\mathcal{B}$ vanishes there.

Therefore, the hollow Skyrmions as stated above are very unstable as $B$ increases. It follows that the true Skyrmions are much more compact as $B$ increases.

Indeed, most true larger Skyrmions are made up of the more stable $B = 4$ cubic units in some arrangement.

## 4.9 Symmetries of rational maps

**Definition:** A rational map $R : S^2 \to S^2$ has a *symmetry* if it is invariant under combined rotations of both the domain *and* target spheres $S^2$.

A rotation $k \in SO(3)$ acts on the $z$-sphere as

$$k(z) = \frac{\gamma z + \delta}{-\bar{\delta} z + \bar{\gamma}},$$

where $|\gamma|^2 + |\delta|^2 = 1$. A rotation $M \in SO(3)$ acts on the $R(z)$-sphere as

$$M(R) = \frac{\Gamma R + \Delta}{-\bar{\Delta} R + \bar{\Gamma}},$$

where $|\Gamma|^2 + |\Delta|^2 = 1$.

A symmetry of $R(z)$ can then be written as $(k, M_k)$ where

$$M_k(R(z)) = R(k(z)).$$

**Definition:** The *symmetry group* of $R(z)$ is the set of all symmetries, together with the group multiplication given by

$$(k_1, M_{k_1}) \cdot (k_2, M_{k_2}) = (k_1 k_2, M_{k_1 k_2}).$$

This makes sense because of the fact that we can evaluate $R(k_1 k_2(z))$ in two ways:

$$M_{k_1} M_{k_2}(R(z)) = R(k_1 k_2(z)), \quad M_{k_1 k_2}(R(z)) = R(k_1 k_2(z)).$$

Hence $M_{k_1 k_2} = M_{k_1} M_{k_2}$, so $M$ is a group homomorphism.

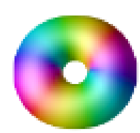
## 4.10 Symmetries of Skyrmions

We can apply the above theory to work out more about symmetries of the rational map approximations to Skyrmions.

Each Skyrmion has a symmetry group of the form $K \leq SO(3) \times SO(3)$, where the first factor of $SO(3)$ is a rotation on the $S^2$ domain sphere, i.e. the $z$-sphere, and the second factor of $SO(3)$ is an *isorotation* of the $R(z)$-sphere.

We associate a *colour* to each of the pion fields $\pi_1$, $\pi_2$, $\pi_3$. In the rational map approximation, this corresponds to where $\hat{\mathbf{n}}_{R(z)}$ is, i.e. where $R(z)$ is on the $R$-sphere.

**Example:** Consider the rational approximation $R(z) = z^2$. This gives a toroidal Skyrmion:



A normal rotation in $z$-space just rotates the Skyrmion around. A rotation in $R(z)$-space rotates the colours around. Thus we spot that a $\pi$ rotation around the hole gives a $2\pi$ rotation of the colours.

This corresponds to (with $\alpha = \pi$) $k(z) = e^{i\alpha}z$ and $R(k(z)) = R(e^{i\alpha}z) = e^{2i\alpha}z^2 = M_k(R(z))$, where $M_k(R) = e^{2i\alpha}R$.

There's another rotation which corresponds to flipping the Skyrmion upside down. The corresponding isorotation also flips the colours around.

In terms of the $z$-sphere, we should view the rotation as $k(z) = 1/z$. This, as expected, gives the same rotation of the colours via

$$M_k(R) = \frac{1}{R}.$$

Finally, one can reflect the Skyrmion, i.e. by $k(z) = \bar{z}$, $M_k(R) = \bar{R}$. Geometrically, this is reflection in the 2-axis.

The full symmetry group is $D_{\infty h}$, where $D_\infty$ is the infinite dihedral group, and $h$ is the reflection symmetry.

---

Note symmetries do *not* leave the Skyrmions invariant on their own. For example, the hedgehog has pion fields: $\boldsymbol{\pi} = \sin(f(r))\hat{\mathbf{x}}$.

Under a rotation of $z$-space, we have $\hat{\mathbf{x}} \mapsto r\hat{\mathbf{x}}$. Under a rotation of $R(z)$-space, we have $\boldsymbol{\pi} \mapsto M(r)\boldsymbol{\pi}$. It is only under the *combined* transformation, where $M(r) = r$, that the Skyrmion is invariant: $M(r)\boldsymbol{\pi} = \sin(f(r))r\hat{\mathbf{x}}$.

---

## 4.11 Quantisation of Skyrmions

We need to construct wavefunctions, which are functions on the classical configuration space.

**Definition:** The Skyrme field *configuration space*, for baryon number $B$, is:

$$\mathcal{C}_B = \text{Maps}_B(\mathbb{R}^3 \to SU(2)) \cong \text{Maps}_B(S^3_\infty \to S^3),$$

where $\text{Maps}_B$ denotes the space of maps of degree $B$.

---

**Theorem:** Topologically,

$$\text{Maps}_B(S^3_\infty \to S^3) \cong \text{Maps}_0(S^3_\infty \to S^3).$$

*Proof:* We can multiply each of the maps in the first set by a fixed field configuration with baryon number $B$. Since Skyrme fields live in a group, we can invert the map; since it is a Lie group, the map is smooth and has a smooth inverse.

Provided that degrees add, the spaces are homeomorphic. We can show this as follows. Suppose $U_1(\mathbf{x})$ and $U_2(\mathbf{x})$ have degrees $B_1$ and $B_2$ separately. Then for large $\mathbf{a}$,

$$U_1(\mathbf{x} - \mathbf{a})U_2(\mathbf{x}),$$

it's clear that the baryons are completely separate from one another. So we get contributions separately from both to get degree $B_1 + B_2$. Taking $\mathbf{a} \to \mathbf{0}$, the degree is conserved since it is topological. $\square$

---

The important fact about the configuration spaces is that they are *connected* but not *simply-connected*. It can be shown that the homotopy group of $\text{Maps}_0(S^3_\infty \to S^3)$ is: $\pi_1(\text{Maps}_0(S^3_\infty \to S^3)) \cong \mathbb{Z}_2$. This has important consequences when we quantise:

- A wavefunction on $\mathcal{C}_B$ is allowed to be multi-valued because of the simply-connected configuration space.
- In this particular case, the *sign* of the wavefunction changes when we move around a non-contractible loop in $\mathcal{C}_B$. This means these particles are *fermionic*.

## 4.12 Rigid body quantisation

We assume that Skyrmions are rigid bodies. Then there are $9$ possible coordinates describing the Skyrmion: (i) 3 translations, (ii) 3 rotations and (iii) 3 isorotations. These describe the orientation of the Skrymions in terms of their (i) position; (ii) orientation; (iii) colouring.

Associated to each of these symmetries, there are Hermitian operators: (i) **P**, momentum, (ii) **J**, spin; (iii) **I**, isospin.

Assume Skyrmion is fixed in space, and hence ignore momentum. Then we can describe the state of any quantised Skyrmion in the form:

$$|\psi\rangle = |J\ L_3\ J_3\rangle \otimes |I\ K_3\ I_3\rangle\,,$$

where $J$ is the *total spin*, $J_3$ is the $3$rd spin component with respect to the space axes and $L_3$ is the $3$rd spin component with respect to the body axes. Similarly for isospin.

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## 4.13 Examples of Skyrmion quantisation

**Example 1:** Consider the $B = 1$ hedgehog solution. For any $r \in SO(3)$ under which $\mathbf{x}$ transforms, the hedgehog is symmetric with corresponding isorotation $M(r) = r$, under which $\boldsymbol{\pi}$ transforms. Hence:

$$e^{i\alpha\hat{\mathbf{n}}\cdot\mathbf{L}}e^{i\alpha\hat{\mathbf{n}}\cdot\mathbf{K}}\,|\psi\rangle = |\psi\rangle\,,$$

for any angle $\alpha$ and rotation axis $\hat{\mathbf{n}}$. We get a $+$ sign on the RHS since we can take $\alpha \to 0$ continuously.

Infinitesimally,

$$(\mathbf{L} + \mathbf{K})\,|\psi\rangle = 0.$$

We call $\mathbf{L} + \mathbf{K}$ *grand spin*, i.e. the sum of the spin and isospin. This result means that the magnitudes of **L** and **K** must be the same, and hence $J = I$.

Later we will see (by the Krusch formula) that:

$$e^{2\pi i\hat{\mathbf{n}}\cdot\mathbf{L}}\,|\psi\rangle = -\,|\psi\rangle\,.$$

This implies the particle has fermionic statistics, and hence has half-integer spin.

For $J = 1/2,\ I = 1/2$, there are two isospin states $I_3 = -1/2,\ 1/2$. These are the *neutron* and *proton*.

For $J = 3/2,\ I = 3/2$, there are 4 isospin states, $I_3 = -3/2,\ -1/2,\ 1/2,\ 3/2$. These are the $\Delta^-, \Delta^0, \Delta^+$ and $\Delta^{++}$ *resonances*.

**Example 2:** Recall that for $B = 2$, a rotation about the $3$-axis in the $z$ space is $k(z) = e^{i\alpha}z$, and the corresponding rotation about the $3$-axis in the $R(z)$ space is $M(R) = e^{2i\alpha}R$. This leads to the constraint:

$$e^{i\alpha L_3} e^{2i\alpha K_3} |\psi\rangle = |\psi\rangle \,.$$

As before, we have the $+$ sign because we can take $\alpha \to 0$.

We also get the condition from the rotation about the $1$-axis by $\pi$:

$$e^{i\pi L_1} e^{i\pi K_1} |\psi\rangle = -|\psi\rangle \,.$$

One can see this from the Krusch formula (see below), or we can prove this intuitively as follows.

Begin by separating the Skyrmion out into two $B = 1$ Skyrmions, albeit of different orientations in terms of colours:

In particular, the $\pi_1$ poles are pointing towards one another in the separated Skyrmions.

Now consider applying the operations $e^{i\pi L_1}$ and $e^{i\pi K_1}$ in succession. For $B = 1$ Skyrmions, isorotations are the same as rotations, so the effect of both of these is just to perform a rotation around the $\pi_1$ axis.

In the first case, both Skyrmions rotate in the same direction by $\pi$, since $e^{i\pi L_1}$ is a physical rotation around the $\pi_1$ body axis. In the second case, the Skyrmions rotate in different directions by $\pi$, since $e^{i\pi K_1}$ is an *isorotation* so is rotating the colours.

Therefore, one Skyrmion rotates by $2\pi$ whilst the other rotates by $0$. It follows this is a non-contractible loop since for $B = 1$ a $2\pi$ rotation is a non-contractible loop by the Krusch formula (see later).

Notice also in this case that (around some appropriate symmetry axis exchanging the $B = 1$ Skyrmions) we have:

$$e^{2\pi i \hat{\mathbf{n}}\cdot\mathbf{L}} |\psi\rangle = |\psi\rangle \,,$$

so we get no sign from the wavefunction under identical particle exchange. Hence the $B = 2$ Skyrmion is bosonic, and so has integral spin.

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Thus, the allowed eigenstates obey $(L_3 + 2K_3)|\psi\rangle = 0$. Therefore $L_3 + 2K_3 = 0$ as eigenvalues.

Restrict to the case $L < 2$, $K < 2$. Then $L < 2$ implies $L_3 = -1, 0, 1$ are allowed values, which gives $K_3 = 2, 0, -2$ respectively. Thus the only solution for $L, K < 2$ is $L_3 = K_3 = 0$.

One possible allowed state is:

$$|J = 1, L_3 = 0\rangle \otimes |I = 0, K_3 = 0\rangle \,.$$

This is called the *deuteron*. It is an isospin $0$ bound state of a proton and a neutron. Another possible allowed state is

$$|J = 0, L_3 = 0\rangle \otimes |I = 1, K_3 = 0\rangle \,.$$

This is called the *dinucleon*. It is a resonance, consisting of two neutrons, two protons or the isospin $1$ combination of a neutron and a proton. It is unstable.

Can we have $I = J = 0$? No. We show this as follows. Notice that we have:

$$e^{i\pi L_1} |J, L_3\rangle = (-1)^{J+L_3} |J, -L_3\rangle \,.$$

We get $-L_3$ on the RHS from the rotation; when we flip the Skyrmion upside down, the new body-fixed $3$-axis points in the opposite direction to the old angular momentum in the $3$-direction.

We get $(-1)^{J+L_3}$ from the following. Recall that $e^{i\pi L_1}$ is a rotation about the $1$-axis, sending $\theta \mapsto \pi - \theta$. Recall that the angular wavefunction is proportional to a spherical harmonic, and hence under the transformation we have:

$$\begin{aligned} Y_J^{L_3}(\theta,\phi) &= N e^{iL_3\phi} P_J^{L_3}(\cos(\theta)) \\ \mapsto N e^{iL_3\phi} P_J^{L_3}(-\cos(\theta)) &= (-1)^{J+L_3} Y_J^{L_3}(\theta,\phi), \end{aligned}$$

by a property of the associated Legendre polynomials $P_J^{L_3}$.

Therefore: $e^{i\pi L_1} e^{i\pi K_1} |J, L_3\rangle \otimes |I, K_3\rangle$

$$= (-1)^{J+I+L_3+K_3} |J, -L_3\rangle \otimes |I, -K_3\rangle \,.$$

If $I = J = 0$, then $L_3 = K_3 = 0$, and hence $e^{i\pi L_1} e^{i\pi K_1} |0,0\rangle \otimes |0,0\rangle = |0,0\rangle \otimes |0,0\rangle$ by the above calculation. But this contradicts the condition $e^{i\pi L_1} e^{i\pi K_1} |\psi\rangle = -|\psi\rangle$. So $I = J = 0$ is not possible.

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## 4.14 The Krusch formula

**Theorem:** Suppose a symmetry gives rise to the quantum constraint:

$$e^{i\theta_2 \hat{\mathbf{n}}_2\cdot\mathbf{L}} e^{i\theta_1 \hat{\mathbf{n}}_1\cdot\mathbf{K}} |\psi\rangle = \chi_{\mathsf{FR}} |\psi\rangle \,,$$

where $\chi_{\mathsf{FR}} = \pm$ is called the *Finkelstein-Rubinstein sign*. Then:

$$\chi_{\mathsf{FR}} = (-1)^{\left\{\frac{B}{2\pi}(B\theta_2 - \theta_1)\right\}}.$$

*Proof:* Not required. □