Part II: Dynamical Systems - Revision

Lectures by John Lister, notes by James Moore

1 Basic definitions

1.1 Equations

We study equations of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ for $\mathbf{x} \in E \subseteq \mathbb{R}^n$, where E is called the *state space*. Since $\mathbf{f}$ has no t-dependence, the equation is called *autonomous*.

Non-autonomous ODEs can be written in standard form by setting $\mathbf{y} = (\mathbf{x}, t) \Rightarrow \dot{\mathbf{y}} = (\mathbf{f}(\mathbf{y}), 1)$. Similarly, higher order ODEs can be written in standard form.

Theorem: For $\mathbf{f}$ Lipschitz, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a unique solution in a neighbourhood of $(\mathbf{x}_0, t_0)$.

Proof: See Analysis II. □

1.2 Orbits and invariant sets

Definition: The solution curve $\mathbf{x}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$ to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is called a *trajectory*, or *orbit* or *integral curve* in the state space.

Definition: The *flow* of the vector field $\mathbf{f}(\mathbf{x})$ is the function $\boldsymbol{\phi}_t(\mathbf{x})$ obeying $\partial_t \boldsymbol{\phi}_t(\mathbf{x}) = \mathbf{f}(\boldsymbol{\phi}_t(\mathbf{x}))$, with $\boldsymbol{\phi}_0(\mathbf{x}) = \mathbf{x}$.

Slogan: $\boldsymbol{\phi}_t(\mathbf{x})$ is our position in state space a time t after we were at position $\mathbf{x}$.

Definition: An *orbit* of a flow through $\mathbf{x}_0$ is the set $\mathcal{O}(\mathbf{x}_0) = \{\boldsymbol{\phi}_t(\mathbf{x}_0) : t \in \mathbb{R}\}$. The *forwards orbit* is $\mathcal{O}^+(\mathbf{x}_0) = \{\boldsymbol{\phi}_t(\mathbf{x}_0) : t > 0\}$ and the *backwards orbit* is $\mathcal{O}^-(\mathbf{x}_0) = \{\boldsymbol{\phi}_t(\mathbf{x}_0) : t < 0\}$.

Definition: A set Λ is called *invariant* if $\mathbf{x}_0 \in \Lambda \Rightarrow \mathcal{O}(\mathbf{x}_0) \subseteq \Lambda$.

Slogan: If we start in Λ, we stay in Λ, both forwards and backwards in time.

1.3 Periodic trajectories

Definition: A point $\mathbf{x}_0$ is a *periodic point* of $\mathbf{f}$ with *period* $T > 0$ if $\boldsymbol{\phi}_T(\mathbf{x}_0) = \mathbf{x}_0$ and $\boldsymbol{\phi}_t(\mathbf{x}_0) \neq \mathbf{x}_0$ for all $0 < t < T$. The set $\mathcal{O}(\mathbf{x}_0)$ is called a *periodic orbit*.

Definition: A *limit cycle* is an isolated periodic orbit (that is, there exists an open set containing the limit cycle, but no other periodic orbits).

1.4 Homoclinic and heteroclinic orbits

Definition: Let $\mathbf{x}_0$ be a fixed point. If $\exists \mathbf{y}$ such that $\mathbf{y} \neq \mathbf{x}_0$ and $\boldsymbol{\phi}_t(\mathbf{y}) \to \mathbf{x}_0$ as $t \to \infty$, and $\boldsymbol{\phi}_t(\mathbf{y}) \to \mathbf{x}_0$ as $t \to -\infty$, then $\mathcal{O}(\mathbf{y})$ is called a *homoclinic orbit*.

Definition: Let $\mathbf{x}_0$ and $\mathbf{x}_1$ be distinct fixed points. If $\exists \mathbf{y}$, not equal to either fixed point, such that $\boldsymbol{\phi}_t(\mathbf{y}) \to \mathbf{x}_0$ as $t \to \infty$, and $\boldsymbol{\phi}_t(\mathbf{y}) \to \mathbf{x}_1$ as $t \to -\infty$, then $\mathcal{O}(\mathbf{y})$ is called a *heteroclinic orbit*.

1.5 Limit sets and properties

Definition: The ω*-limit set* of $\mathbf{x}$ is:

$$\omega(\mathbf{x}) = \{\mathbf{y} : \exists (t_n) \text{ with } \boldsymbol{\phi}_{t_n}(\mathbf{x}) \to \mathbf{y}, t_n \to \infty, \text{ as } n \to \infty\}.$$

Definition: The α*-limit set* of $\mathbf{x}$ is:

$$\alpha(\mathbf{x}) = \{\mathbf{y} : \exists (t_n) \text{ with } \boldsymbol{\phi}_{t_n}(\mathbf{x}) \to \mathbf{y}, t_n \to -\infty, \text{ as } n \to \infty\}.$$

Theorem: The following hold:

(1) $\omega(\mathbf{x})$ and $\alpha(\mathbf{x})$ are invariant sets;

(2) If $\mathcal{O}^+(\mathbf{x})$ is bounded, then $\omega(\mathbf{x})$ is non-empty.

Proof: (1). Let $\mathbf{y} \in \omega(\mathbf{x})$. Then $\exists t_n$ with $\boldsymbol{\phi}_{t_n}(\mathbf{x}) \to \mathbf{y}$, $t_n \to \infty$ as $n \to \infty$. Let $T \in \mathbb{R}$. Then by continuity of the flow map, we have $\boldsymbol{\phi}_{t_n+T}(\mathbf{x}) \to \boldsymbol{\phi}_T(\mathbf{y})$, $t_n + T \to \infty$ as $n \to \infty$. Hence $\mathcal{O}(\mathbf{y}) \subseteq \omega(\mathbf{x})$, so $\omega(\mathbf{x})$ is invariant. Similar for $\alpha(\mathbf{x})$.

(2). Suppose $\mathcal{O}^+(\mathbf{x})$ is bounded. Let $\boldsymbol{\phi}_n(\mathbf{x})$ be a sequence in $\mathcal{O}^+(\mathbf{x})$. By the Bolzano-Weierstrass Theorem, there is a convergent subsequence t_n, with $t_n \geq n \Rightarrow t_n \to \infty$ as $n \to \infty$. So $\omega(\mathbf{x})$ is non-empty. □

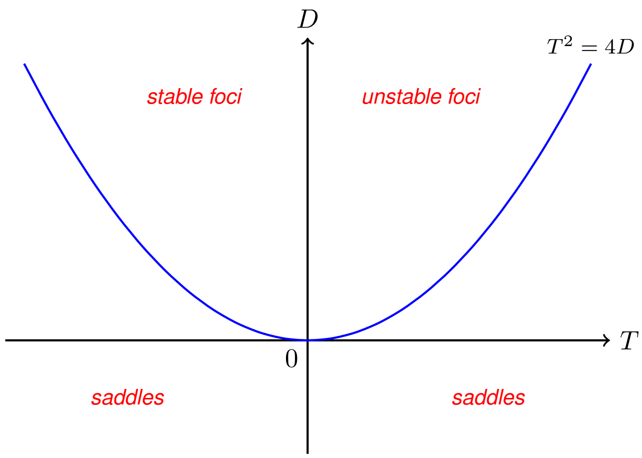
2 Linearisation

2.1 Fixed point classification

Near a fixed point $\mathbf{x} = \mathbf{x}_0$, we can expand $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ to get $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + O(|\mathbf{y}|^2)$, where $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$, and $\mathbf{A}$ is the *Jacobian matrix* given by:

$$A_{ij} = \frac{\partial f_i}{\partial f_j}.$$

The eigenvalues of A satisfy $\lambda^2 - T\lambda + D = 0 \Rightarrow \lambda = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$, where $T = \mathrm{tr}(A)$ and $D = \det(A)$. This gives the classification of fixed points shown in the diagram below.



Definition: *Sources* are fixed points in the first quadrant ('arrows come out'). *Sinks* are fixed points in the second quadrant ('arrows go in'). *Saddles* are fixed points in the lower half plane ('arrows go in and out').

2.2 Notes on the classification

- For a stable node, need $D > 0$, $T^2 > 4D$ and $T < 0$, hence $\lambda_1 < \lambda_2 < 0$, WLOG. Linearisation gives $y_1 = e^{\lambda_1 t}, y_2 = e^{\lambda_2 t} \Rightarrow y_1/y_2 = e^{(\lambda_1 - \lambda_2)t} \to 0$ as $t \to \infty$. So y_2 dominates, and we collapse tangent to the y_2 axis.

- For a *focus*, first find the *nullclines* and then consider signs to get the direction of the spiral.

- *Improper nodes* occur when A is non-diagonalisable, and has eigenvalue not equal to 0. Get equations $\dot{y_2} = \lambda y_2$ and $\dot{y_1} = \lambda y_1 + y_2 = \lambda y_1 + e^{\lambda t}$, so get resonant forcing, and $y_2 \sim te^{\lambda t}$. Also note that to get the Jordan normal form of A in this case, we need generalised eigenvectors (those satisfying $(A - \lambda I)^n \mathbf{v} = \mathbf{0}$.

- *Lines of fixed points* come in two varieties; some have trajectories parallel to the line of fixed point, others have all trajectories converging onto the line of fixed points.

2.3 When is linearisation possible?

From the diagram, it is clear that a small perturbation to a linear system causes only *centres* and *lines of fixed points* to change their topology (e.g. stellar nodes just screw up or bend out instead). So linearisation around centres and lines of fixed points is *unreliable*.

We notice that both these cases have an eigenvalue with $\mathrm{Re}(\lambda) = 0$, and are the only cases with this feature.

Definition: A *hyperbolic fixed point* is a fixed point where no eigenvalue satisfies $\mathrm{Re}(\lambda) = 0$. Else, a fixed point is *non-hyperbolic*.

Linearisation is only reliable for *hyperbolic fixed points*.

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3 The Stable Manifold Theorem

Definition: The *stable, unstable and centre linear invariant subspaces* of the linearisation of $\mathbf{f}$ at a fixed point $\mathbf{x}_0$ are the local linear subspaces spanned by the subsets of eigenvectors of A (possible generalised) satisfying $\mathrm{Re}(\lambda) < 0$, $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Re}(\lambda) = 0$ respectively. They are denoted E^S, E^U and E^C respectively.

Theorem (The Stable Manifold Theorem): Let $\mathbf{0}$ be a hyperbolic fixed point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with linear invariant subspace E^S and E^U ($E^C = \emptyset$ since hyperbolic). Then in some open neighbourhood N of the origin $\mathbf{0}$, there exists *local stable and unstable manifolds*:

$$W^S_{\mathrm{loc}} = \{\mathbf{x} : \boldsymbol{\phi}_t(\mathbf{x}) \to \mathbf{0} \text{ as } t \to \infty\} \cap N;$$
$$W^U_{\mathrm{loc}} = \{\mathbf{x} : \boldsymbol{\phi}_t(\mathbf{x}) \to \mathbf{0} \text{ as } t \to -\infty\} \cap N,$$

satisfying: (i) these have the same dimensions as E^S and E^U respectively; (ii) these are tangent to E^S and E^U at $\mathbf{0}$ respectively.

We use the Stable Manifold Theorem to get the behaviour of trajectories slightly further from the fixed point.

Example: Let $\dot{x} = x - xy$ and $\dot{y} = -y + x^2$. Then E^S is $x = 0$ and E^U is $y = 0$. Let the stable manifold by $x = S(y)$.

By the Theorem, need $S(0) = S'(0) = 0$. So expand as $x = S(y) = a_2 y^2 + a_3 y^3 \ldots$. Substitute into the chain rule:

$$\dot{x} = \frac{dS}{dy}\dot{y} \qquad \Rightarrow \qquad x - xy = S'(y)(-y + x^2)$$

and insert the series to find the coefficients.

4 Hamiltonian Systems

Linearisation around non-hyperbolic fixed points is unreliable - we need more information to determine the behaviour, e.g.

Definition: A system is *Hamiltonian* if $\exists H$ such that $\dot{x} = \partial_y H$, $\dot{y} = -\partial_x H$.

Theorem: Hamiltonian systems satisfy:

(1) All fixed points are centres or saddles.

(2) Trajectories are lines of constant H.

(3) $\nabla \cdot \mathbf{f} = 0$.

Proof: (1). Jacobian matrix has trace $H_{yx} - H_{xy} = 0$ hence centres or saddles only.

(2). Lines of constant H: $0 = \dot{H} = \dot{x} \cdot \nabla H$. Hence trajectories move orthogonal to $\nabla H \Rightarrow$ they move parallel to contours of H.

(3). $\partial_x(\partial_y H) - \partial_y(\partial_x H) = 0$. $\square$

We can thus use Hamiltonian systems to detect nonlinear centres, say, by finding closed contours around a fixed point.

Example: Consider $\dot{x} = y$ and $\dot{y} = -x + x^2$. This is Hamiltonian with $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$.

Hence contours of constant H look like the square root of cubic curves. Sketching, we have:

5 Stability of invariant sets

5.1 Stability of fixed points

Definition: A fixed point $\mathbf{x}_0$ is *Lypanunov stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|\mathbf{x} - \mathbf{x}_0| < \delta \Rightarrow |\boldsymbol{\phi}_t(\mathbf{x}) - \mathbf{x}_0| < \epsilon$ for all $t > 0$.

Slogan: 'If we start close enough, we'll stay close.'

Definition: A fixed point $\mathbf{x}_0$ is *quasi-asymptotically stable* if $\exists \delta > 0$ such that $|\mathbf{x} - \mathbf{x}_0| < \delta \Rightarrow \boldsymbol{\phi}_t(\mathbf{x}) \to \mathbf{x}_0$ as $t \to \infty$.

Slogan: 'If we start close enough, we'll get there eventually.'

Definition: A fixed point $\mathbf{x}_0$ is *asymptotically stable* if it is both Lyapunov and quasi-asymptotically stable.

5.2 Stability of invariant sets

Definition: The δ-*neighbourhood* of the invariant set Λ is $N_\delta(\Lambda) = \{\mathbf{x} : \exists \mathbf{y} \in \Lambda \text{ with } |\mathbf{x} - \mathbf{y}| < \delta\}$.

Definition: We say $\boldsymbol{\phi}_t(\mathbf{x}) \to \Lambda$ as $t \to \infty$ if $\inf_{\mathbf{y} \in \Lambda} |\boldsymbol{\phi}_t(\mathbf{x}) - \mathbf{y}| \to \mathbf{0}$ as $t \to \infty$.

We get the stability definitions for invariant sets from fixed points by promoting to δ/ϵ-neighbourhoods and using the limit defined above. So:

Definition: An invariant set Λ is *Lyapunov stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\mathbf{x} \in N_\delta(\Lambda) \Rightarrow \boldsymbol{\phi}_t(\mathbf{x}) \in N_\epsilon(\Lambda)$ for all $t > 0$.

Definition: An invariant set Λ is *quasi-asymptotically stable* if $\exists \delta > 0$ such that $\mathbf{x} \in N_\delta(\Lambda) \Rightarrow \boldsymbol{\phi}_t(\mathbf{x}) \to \Lambda$ as $t \to \infty$.

Definition: An invariant set Λ is *asymptotically stable* if it is both Lyapunov and quasi-asymptotically stable.

Definition: The *domain of stability* of an asymptotically stable set Λ is $\{\mathbf{x} : \boldsymbol{\phi}_t(\mathbf{x}) \to \Lambda \text{ as } t \to \infty\}$. If this is the whole state space, Λ is *globally stable*.

6 Lyapunov and bounding functions

6.1 Lyapunov functions

Definition: A *Lyapunov function for* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ for which, on some domain D containing the origin, we have:

(a) $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ for all $\mathbf{x} \in D$;

(b) $\dot{V} \leq 0$ for all $\mathbf{x} \in D$.

Idea: At any point in the domain D, need $\dot{V} \leq 0$; in particular, need $\dot{V} \leq 0$ on any trajectory. So all trajectories head 'downhill'.

Theorem (Lyapunov's First Theorem): If a Lyapunov function exists, $\mathbf{0}$ is Lyapunov stable.

Proof: Let $\epsilon > 0$ and assume WLOG that it is small enough for $\{\mathbf{x} : |\mathbf{x}| \leq \epsilon\} \subseteq D$.

Define $m = \inf\{V(\mathbf{x}) : |\mathbf{x}| = \epsilon\}$. Because $|\mathbf{x}| = \epsilon$ is compact, the infimum is attained for some $\mathbf{x}_0$ such that $|\mathbf{x}_0| = \epsilon$; so $0 < V(\mathbf{x}_0) = m$.

Define $C = \{\mathbf{x} : V(\mathbf{x}) < m\} \cap \{\mathbf{x} : |\mathbf{x}| < \epsilon\}$. C certainly contains $\mathbf{0}$ since $V(\mathbf{0}) = 0 < m$; and hence it also contains a small open neighbourhood containing zero. Also, if $\mathbf{x} \in C$, then $\boldsymbol{\phi}_t(\mathbf{x}) \in C$ for all $t > 0$, since $V \geq m$ outside C.

Hence take any δ such that $\{\mathbf{x} : |\mathbf{x}| < \delta\} \subset C$ and we're done. □

Theorem (La Salle's Invariance Principle): Let V be a Lyapunov function on a bounded domain D, and let $\mathcal{O}^+(\mathbf{x}) \subseteq D$. Then $\boldsymbol{\phi}_t(\mathbf{x})$ tends to an invariant subset of $\{\mathbf{x} : \dot{V} = 0\} \cap D$.

Proof: (1). D is bounded, so $\omega(\mathbf{x})$ is non-empty. Thus, by definition, $\boldsymbol{\phi}_t(\mathbf{x}) \to \omega(\mathbf{x})$ as $t \to \infty$. So remainder of proof: show that $\omega(\mathbf{x})$ is an invariant set (which we know already) and a subset of $\{\mathbf{x} : \dot{V} = 0\} \cap D$.

(2). Show that $V(\boldsymbol{\phi}_t(\mathbf{x}))$ tends to a limit. Since $\dot{V} < 0$ in D and $V \geq 0$, $V(\boldsymbol{\phi}_t(\mathbf{x}))$ is monotonically decreasing (since $\mathcal{O}^+(\mathbf{x}) \subseteq D \Rightarrow \boldsymbol{\phi}_t(\mathbf{x}) \in D$ for all $t > 0$) and bounded below. So tends to a limit $\alpha \geq 0$.

(3). Let $\mathbf{y} \in \omega(\mathbf{x})$. Then $\exists t_n$ such that $\boldsymbol{\phi}_{t_n}(\mathbf{x}) \to \mathbf{y}$ as $n \to \infty$, hence $V(\mathbf{y}) = \alpha$ by continuity of V.

(4). Finally, since $\omega(\mathbf{x})$ is an invariant set, $\boldsymbol{\phi}_t(\mathbf{y}) \in \omega(\mathbf{x})$, and so $V(\boldsymbol{\phi}_t(\mathbf{y})) = \alpha$ for all t; thus $\dot{V}(\mathbf{y}) = 0$, and we're done.

Corollary: Let V be Lyapunov on a bounded domain D, and let the only invariant subset of $\{\mathbf{x} : \dot{V} = 0\} \cap D$ be $\{\mathbf{0}\}$. Then $\mathbf{0}$ is asymptotically stable.

Proof: By Lyapunov's First Theorem, it's Lyapunov stable.

Let k be such that $C = \{\mathbf{x} : V(\mathbf{x}) < k\} \subseteq D$ and let δ be such that $\{\mathbf{x} : |\mathbf{x}| < \delta\} \subseteq C$. Trajectories starting in C cannot escape, so La Salle's Invariance Principle applies to C. It follows that trajectories starting in $\{\mathbf{x} : |\mathbf{x}| < \delta\}$ must tend to $\mathbf{0}$, giving quasi-asymptotic stability, and hence asymptotic stability. □

6.2 Example use of Lyapunov function

Consider $\dot{x} = -x + y^2 x$ and $\dot{y} = -y + x^2 y$. Try $V = x^2 + b^2 y^2$, so that $\dot{V} = -2(x^2 + b^2 y^2) + 2(1 + b^2) x^2 y^2$.

Writing $(x, by) = \sqrt{V}(\cos(\phi), \sin(\phi))$ gives

$$\dot{V} = -2V + \frac{2(1 + b^2) V^2 \cos^2(\phi) \sin^2(\phi)}{b^2}.$$

But $\cos^2(\phi) \sin^2(\phi) \leq 1/4$, and hence $\dot{V} \leq 0$ for $0 < V < 4b^2/(1 + b^2)$.

Varying b implies that V is Lyapunov on various ellipses; by changing b we get a larger estimate for the domain of stability.

6.3 Bounding functions

Definition: A *bounding function* is a function $V(\mathbf{x})$ with bounded contours, which satisfies $\dot{V} < -\delta < 0$ for some $\delta > 0$ for $V \geq M$.

Idea: All trajectories eventually enter $\{V(\mathbf{x}) \leq M\}$ and stay there.

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7 Existence of periodic orbits

7.1 The Poincaré index test

Definition: Let Γ be a simple closed curve in $\mathbb{R}^2$ passing through no fixed points, and let $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$. Define the angle $\psi(\mathbf{x}) = \arctan(f_2/f_1)$ everywhere on the curve. The *Poincaré index* of Γ, denoted $I(\Gamma)$ is the integer multiple of 2π by which ψ changes as we move anticlockwise around Γ once.

Theorem: The Poincaré index has the properties:

(a)
$$I(\Gamma) = \frac{1}{2\pi} \oint_\Gamma d\psi = \frac{1}{2\pi} \oint_\Gamma \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}.$$

(b) $I(\Gamma)$ is unaffected by continuous deformation of Γ not passing through a fixed point.

(c) If Γ encloses no fixed points, $I(\Gamma) = 0$.

(d) The Poincaré index is additive in the sense: $I(\Gamma_1) = I(\Gamma_2) + I(\Gamma_3)$, as in the below picture.

Proof: (a) by Definition.

(b) follows since the integral in (1) must be integer-valued, but must also have continuous dependence on Γ provided $f_1^2 + f_2^2 \neq 0$ (so no jumps allowed in $I(\Gamma)$ except when passing over a fixed point, i.e. $f_1 = f_2 = 0$).

(c) - use (b) to contract to a single point, where ψ is constant.

(d) - by picture below.

Definition: The *Poincaré index of a fixed point* is the Poincaré index of any closed curve encircling the fixed point, and no others.

This is well-defined by (2) in the Theorem above, and (4) show that $I(\Gamma)$ is the sum of the Poincaré indices of the fixed points Γ encloses.

The important Poincaré indices are:

- Periodic orbits have index $+1$.
- Centres, nodes and foci have index $+1$.
- Saddles have index -1.

This gives rise to:

Theorem (The Poincaré index test): There are no periodic orbits of a dynamical system if it is impossible to encircle a subset of the fixed points so that their Poincaré indices sum to $+1$.

Proof: Suppose that there is a periodic orbit. Its Poincaré index must be $+1$, so must encircle fixed points with indices summing to $+1$. Contradiction. □

7.2 Dulac's criterion

Theorem (Dulac's criterion): Let ϕ be a continuously differentiable function such that $\nabla \cdot (\phi \mathbf{f}) \neq 0$ on a simply-connected domain D. Then there are no periodic orbits of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in D.

Proof: Since ϕ is continuously differentiable and so is $\mathbf{f}$, we have that $\phi\mathbf{f}$ is continuously differentiable. Hence $\nabla \cdot (\phi \mathbf{f}) \neq 0$ is single-signed on D, as it can't cross zero without a jump. WLOG $\nabla \cdot (\phi \mathbf{f}) > 0$, else take $-\phi$.

Let Γ be a periodic orbit in D. Then $\mathbf{f}$ is tangent to Γ everywhere. If the orbit encircles the area A and its outward normal is $\hat{\mathbf{n}}$, we have:

$$0 = \oint_\Gamma \phi \mathbf{f} \cdot \hat{\mathbf{n}} ds = \iint_A \nabla \cdot (\phi \mathbf{f}) dA > 0,$$

by 2D divergence theorem. Contradiction. □

Dulac's criterion can also be extended to multiply-connected domains. For a doubly-connected domain, it says there is *at most* one periodic orbit.

You must also be aware of the topology of the state space when using Dulac's criterion. E.g. for a cylindrical topology, an application of Dulac would tell us that there was at most one periodic orbit, and that it encircled the cylinder (since it is a doubly-connected domain).

7.3 The Poincaré-Bendixson Theorem

Theorem: Let $\mathcal{O}^+(\mathbf{x})$ remain in a compact, multiply-connected set $K \subseteq \mathbb{R}^2$ that contains no fixed points. Then $\omega(\mathbf{x})$ is a periodic orbit.

Proof: Non-examinable. □

Example: Consider $\dot{r} = r(1 - r^2(1 + \cos^2(\theta)))$ and $\dot{\theta} = 1 + \frac{1}{2}r^2\sin(2\theta)$. Then $r(1-2r^2) \leq \dot{r} \leq r(1-r^2)$.

So trajectories entering the annulus $K = \{\frac{1}{\sqrt{2}} \leq r \leq 1\}$ cannot leave it. K is compact since it is closed and bounded. We can easily check there are no fixed points in K, and hence there is a periodic orbit by the Poincaré-Bendixson Theorem.

7.4 Nearly-Hamiltonian flows

Definition: Systems of the form:

$$\dot{x} = f_1 + \epsilon g_1,$$
$$\dot{y} = f_2 + \epsilon g_2,$$

with ϵ very small, and the system with $\epsilon = 0$ Hamiltonian (i.e. $f_1 = \frac{\partial H}{\partial y}$, $f_2 = -\frac{\partial H}{\partial x}$), are called *nearly-Hamiltonian systems*.

We can find the periodic orbits of a nearly-Hamiltonian system using the *energy-balance method*:

(1) Find the Hamiltonian H of the system with $\epsilon = 0$.

(2) On a periodic orbit Γ of the full system with $\epsilon \neq 0$, we clearly must have:

$$0 = \oint_\Gamma dH = \oint_\Gamma \dot{H}dt = \epsilon \oint_\Gamma (g_2 f_1 - g_1 f_2)dt.$$

(3) For ϵ small, we can reasonably approximate Γ by a closed contour of H, say $H = H_0$. We assume this closed contour is a distance $O(\epsilon)$ from the true periodic orbit. Then:

$$0 \approx \epsilon \oint_{H=H_0} (g_2 f_1 - g_1 f_2)dt + O(\epsilon^2).$$

(4) This gives an equation of the form $0 = F(H_0)$. Solving gives approximations to the periodic orbits, if there are any.

Note that simple roots of the equation $0 = F(H_0)$ *always* give rise to periodic orbits, but nothing can be inferred from double roots of the equation. See the diagram.

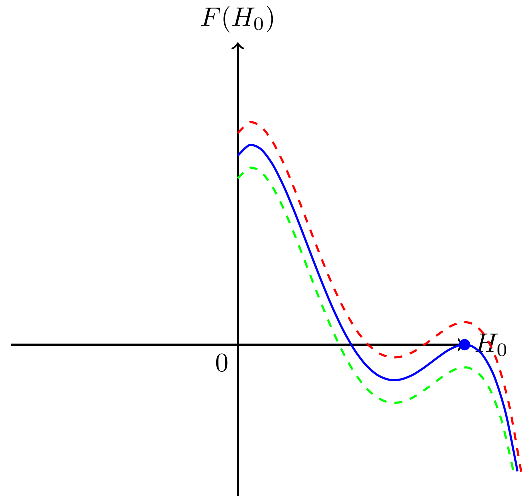

Figure 1: *Blue: the approximation to $F(H_0)$ - note the double root. The true solution may have no periodic orbits near the double root (green) or two (red).*

Example: Consider $\dot{x} = p$ and $\dot{p} = \epsilon(x-3)(x+1)p - x$. Then $H = \frac{1}{2}p^2 + \frac{1}{2}x^2$. The contours $H = H_0$ can be parametrised as $x = \sqrt{2H_0}\cos(t)$, $p = -\sqrt{2H_0}\sin(t)$.

We have $\dot{H} = \epsilon(x-3)(x+1)p^2$. So energy-balance implies:

$$0 = 2\epsilon H_0 \int_0^{2\pi} (\sqrt{2H_0}\cos(t) - 3)(\sqrt{2H_0}\cos(t) + 1)\sin^2(t)dt$$
$$= ... = \epsilon H_0 \pi (H_0 - 6).$$

Thus $p^2 + x^2 = 12$ is an approximation to the periodic orbit of the system.

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8 Stability of periodic orbits

8.1 Floquet theory

Let $\mathbf{x} = \mathbf{X}(t)$ be a periodic orbit with $\mathbf{X}(0) = \mathbf{X}(T) = \mathbf{X}_0$. Perturb the orbit: $\mathbf{x} = \mathbf{X}(t) + \boldsymbol{\eta}(t)$. Substituting into $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ yields:

$$\dot{\boldsymbol{\eta}} = (\boldsymbol{\eta}\cdot\nabla)\mathbf{f}(\mathbf{X}) + O(|\boldsymbol{\eta}|^2) \qquad \Rightarrow \qquad \dot{\boldsymbol{\eta}} = A\boldsymbol{\eta},$$

where A is the Jacobian matrix.

Since the solutions of the linear equation $\dot{\boldsymbol{\eta}} = A\boldsymbol{\eta}$ must be linear combinations of the initial conditions, we can write $\boldsymbol{\eta}(t) = \Phi(t)\boldsymbol{\eta}(0)$ for some matrix $\Phi(t)$.

Definition: $\Phi(t)$ is called the *Floquet matrix* of the periodic orbit.

Theorem: The Floquet matrix obeys the equations:

$$\dot{\Phi} = A\Phi, \qquad \Phi(0) = I.$$

Proof: By definition, $\boldsymbol{\eta}(0) = \Phi(0)\boldsymbol{\eta}(0)$. But $\boldsymbol{\eta}$ is some arbitrary perturbation, hence $\Phi(0) = I$.

For the first equation, substitute $\boldsymbol{\eta}(t) = \Phi(t)\boldsymbol{\eta}(0)$ into $\dot{\boldsymbol{\eta}} = A\boldsymbol{\eta}$ to get $\dot{\Phi}\boldsymbol{\eta}(0) = A\Phi\boldsymbol{\eta}(0)$. Again since $\boldsymbol{\eta}(0)$ is arbitrary, we get the first equation. □

Theorem: $\boldsymbol{\eta}(nT) = (\Phi(T))^n\boldsymbol{\eta}(0)$.

Proof: Since A is periodic with period T, $\Phi(t)$ is periodic with period T. Thus $\boldsymbol{\eta}(nT) = \Phi(T)\boldsymbol{\eta}((n-1)T) = ... = (\Phi(T))^n\boldsymbol{\eta}(0)$. □

The above Theorem shows that whether a perturbation grows or not depends on the size of the eigenvalues of the Floquet matrix relative to 1.

Definition: The eigenvalues of the Floquet matrix at $t = T$, $\Phi(T)$, are called the *Floquet multipliers*.

Since a perturbation *around* the periodic orbit itself does nothing, one Floquet multiplier must always be 1.

Definition: The Floquet multiplier equal to 1 is called the *trivial Floquet multiplier*.

Definition: If all non-trivial Floquet multipliers satisfy $|\lambda| > 1$, the orbit is called *unstable*; if all non-trivial Floquet multipliers satisfy $|\lambda| < 1$, the orbit is called *stable*.

Definition: If one non-trivial Floquet multiplier satisfies $|\lambda| = 1$, the orbit is called *non-hyperbolic*, else it is called *hyperbolic*.

Hyperbolic periodic orbits are structurally stable to small perturbations (compare with linearisation around a hyperbolic fixed point).

Theorem: $\det(\Phi)$ satisfies

$$\frac{d}{dt}(\det(\Phi)) = (\nabla \cdot \mathbf{f}) \det(\Phi).$$

Proof (in 2D): Let ϵ_{ij} denote the Levi-Civita symbol. Then $\det(\Phi) = \epsilon_{ij}\Phi_{1i}\Phi_{2j}$ (summation convention applies). Differentiating:

$$\begin{aligned}\partial_t(\det(\Phi)) &= \epsilon_{ij}(\dot{\Phi}_{1i}\Phi_{2j} + \Phi_{1i}\dot{\Phi}_{2j}) \\ &= \epsilon_{ij}(A_{1k}\Phi_{ki}\Phi_{2j} + A_{2k}\Phi_{1i}\Phi_{kj}).\end{aligned}$$

Since the determinant of a matrix with two identical columns is zero, this reduces to:

$$\epsilon_{ij}(A_{11}\Phi_{1i}\Phi_{2j} + A_{22}\Phi_{1i}\Phi_{2j}) = (\nabla \cdot \mathbf{f}) \det(\Phi),$$

as required. □

Definition: In $\mathbb{R}^2$, the non-trivial Floquet multiplier is given by:

$$\exp\left(\oint_{t=0}^{T} \nabla \cdot \mathbf{f}\, dt\right),$$

where the integral is over the orbit.

Proof: Since the determinant of $\Phi(T)$ is the product of the Floquet multipliers, and the trivial multiplier is 1. Simply using the Theorem above, and $\det(\Phi(0)) = I$, the result follows. □

8.2 The Van der Pol oscillator

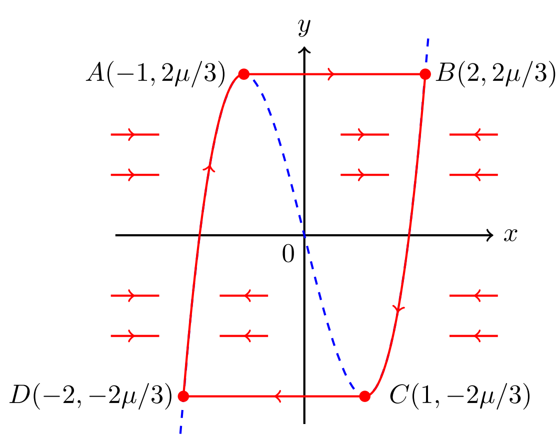
The Van der Pol oscillator has the equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

which can be written as the system

$$\dot{x} = y - \mu x\left(\frac{x^2}{3} - 1\right), \qquad \dot{y} = -x.$$

For μ very large, $\dot{x}$ is huge compared to $\dot{y}$, *unless* $y \approx \mu x(x^2/3 - 1)$. So we get a phase portrait that looks like the diagram below, showing that we get a periodic orbit.



Along $A \to B$ and $C \to D$, the time taken is of order $\Delta t \sim \Delta y/\dot{y} \sim O(\mu)$ (i.e. large), whilst along $B \to C$, $D \to A$, the time taken is of order $\Delta t \sim \Delta x/\dot{x} \sim O(1/\mu)$ (i.e. small). So period is approximately:

$$T \approx 2\int_A^B dt = 2\int_A^B \frac{1}{\dot{y}}\frac{dy}{dx}dx = 2\int_{-2}^{-1} \frac{\mu(x^2-1)}{-x}dx.$$

9 Bifurcation theory

9.1 Definition of a bifurcation

Definition: Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \mu)$ where μ is some parameter. A *bifurcation* is a change in the topological structure of the flows of the dynamical system as the parameter μ passes through a critical value μ_0, which we call a *bifurcation point*.

9.2 Centre manifold theory

Theorem (The Centre Manifold Theorem): Let $\mathbf{0}$ be a non-hyperbolic fixed point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with stable, unstable and centre subspaces E^S, E^U and E^C respectively. Then there exist stable, unstable and centre manifolds W^S, W^U and W^C respectively, with the same dimensions as their respective subspaces, and which are tangent to their respective subspaces at $\mathbf{0}$. The manifolds are themselves invariant sets.

Using the Centre Manifold Theorem, we can classify bifurcations via the following method:

(1) Identify the value of μ for which a bifurcation occurs, μ_0, and the fixed point (x_0, y_0) at which it occurs.

(2) Translate the system via $X = x - x_0$, $Y = y - y_0$ and $\mu' = \mu - \mu_0$ so that the bifurcation occurs at the origin of the extended system (X, Y, μ).

(3) Linearise the extended system as:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{\mu}' \end{pmatrix} = A \begin{pmatrix} X \\ Y \\ \mu \end{pmatrix} + \text{ higher order terms.}$$

Find the zero (generalised) eigenvectors of A, $\mathbf{e}_1$ and $\mathbf{e}_2$. Compute $\mathbf{e}_1 \times \mathbf{e}_2$, which is normal to the tangent plane to the centre manifold at the origin.

(4) To get more terms, series expand, writing $aX + bY + c\mu' = a_{20}X^2 + a_{11}X\mu' + a_{02}(\mu')^2...$. Determine the coefficients using the chain rule:

$$\dot{Y} = \frac{\partial Y}{\partial X}\dot{X}.$$

(Note that $\dot{\mu}' = 0$, so no need for second term in chain rule.)

(5) Replace Y by the series expansion in X and μ' in the equation for $\dot{X}$ to get the evolution equation on the centre manifold. From this, the type of bifurcation can be inferred.

9.3 Example using centre manifold theory

Consider $\dot{x} = x(1 - y - 4x^2)$, $\dot{y} = y(\mu - y - x^2)$. The fixed point $(0, \mu)$ gives a Jacobian with determinant $\mu(\mu - 1)$, so there are bifurcations at $\mu = 0$ and $\mu = 1$. We'll analyse the one at $\mu = 1$.

Begin by defining $X = x$, $Y = y - \mu$, and $\mu' = \mu - 1$. Then the linearised extended system is:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{\mu}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \mu' \end{pmatrix}.$$

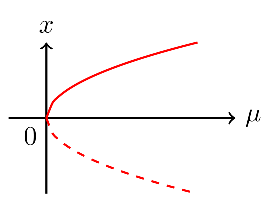
Hence the zero eigenvectors are $(1, 0, 0)^T$ and $(0, 0, 1)^T$, from which it follows that $Y = 0$ is the first approximation to the centre manifold. More terms are possible via series expansion.

9.4 Classification of stationary bifurcations

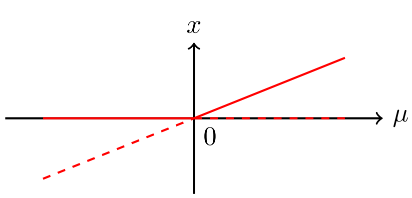
Definition: A *stationary bifurcation* occurs when one of the eigenvalues of a fixed point passes through $\mathrm{Re}(\lambda) = 0$.

All of the simple stationary bifurcations in this course can be reduced to one of the following forms:

(1) **Saddle-node bifurcations:** $\dot{x} = \mu - x^2$.



(2) **Transcritical bifurcations:** $\dot{x} = \mu x - x^2$.



(3) **Pitchfork bifurcations:** $\dot{x} = \mu x - a x^3$. Called supercritical if $a > 0$, subcritical if $a < 0$.

9.5 Reduction to normal form

Often it won't be immediately obvious which bifurcation we have. We massage the equation

$$\dot{x} = f(x, \mu) = \nu_0(\mu) + \nu_1(\mu)x + \frac{1}{2}\nu_2(\mu)x^2 + ...,$$

where $\nu_0(0) = \nu_1(0) = 0$ (to ensure non-hyperbolic fixed point) until we reduce to a normal form.

(1) If $\nu_0'(0) \neq 0$ (i.e. there is a term linear in μ) and $\nu_2(0) \neq 0$, then

$$\dot{x} = \nu_0 + \nu_1 x + \frac{1}{2}\nu_2 x^2 + ...$$

We then rescale time $T = -\frac{\nu_2 t}{2}$ and change the origin by defining $X = x + \nu_1/\nu_2$. The normal form for a saddle-node bifurcation follows.

(2) If $\nu_2(0) \neq 0$ but $\nu_0 \equiv 0$, because the origin is always a fixed point, then

$$\dot{x} = \nu_1 x + \frac{1}{2}\nu_2 x^2 + ...$$

We now rescale $T = -\frac{\nu_2 t}{2}$. Then the normal form for a transcritical bifurcation follows.

(3) If $\nu_0 \equiv \nu_2 \equiv 0$ due to a symmetry, then

$$\dot{x} = \nu_1 x + \frac{1}{6}\nu_3 x^3 + ...,$$

which gives the normal form for a pitchfork bifurcation after rescaling the variables in the same manner.

Despite all this, a better way to classify the fixed points is just to *sketch the bifurcation diagram*.

9.6 Stability of bifurcations

Consider adding a small positive constant ϵ to the equation for $\dot{x}$, e.g. $\dot{x} = \epsilon + \mu x - x^2$. Addition of the constant causes:

- unstable branches to move down the bifurcation diagram;
- stable branches to move up the bifurcation diagram.

Thus we see that the only structurally stable bifurcation is the saddle-node bifurcation.

Again, the best way to check this is to *sketch the bifurcation diagram*.

9.7 Hopf bifurcations

Definition: A *Hopf bifurcation* occurs when a pair of complex eigenvalues cross $\mathrm{Re}(\lambda) = 0$ at $\mu = 0$.

The normal form for a Hopf bifurcation is:

$$\dot{r} = r(\mu - ar^2) + O(r^5),$$
$$\dot{\theta} = \omega + \mu c - br^2 + O(r^4).$$

where $a > 0$ is the supercritical case, and $a < 0$ is the subcritical case. The origin is a fixed point with eigenvalues $\lambda = \mu \pm i(\omega + \mu c)$.

Hence, a Hopf bifurcation is:

- A transition from a stable focus to a stable limit cycle containing an unstable focus, in the *supercritical* case.
- A transition from an unstable limit cycle containing a stable focus, to an unstable focus, in the *subcritical* case.

The bifurcation diagrams look like:

10 Bifurcations in maps

10.1 Definitions of properties for maps

Definition: A *fixed point* of a map **F** is a point $\mathbf{x}_0$ such that $\mathbf{F}(\mathbf{x}_0) = \mathbf{x}_0$.

Definition: An N*-cycle* of a map **F** is an N-tuple $(\mathbf{x}_0, \mathbf{x}_1, ...\mathbf{x}_{N-1})$ with $\mathbf{x}_1 = \mathbf{F}(\mathbf{x}_0)$, $\mathbf{x}_2 = \mathbf{F}(\mathbf{x}_1)$ ... and $\mathbf{x}_0 = \mathbf{F}(\mathbf{x}_{N-1}) = \mathbf{F}^N(\mathbf{x}_0)$, provided that $\mathbf{x}_i \neq \mathbf{x}_0$ for $i = 1, 2, ...N-1$.

Definition: A set A is called *invariant* if $\mathbf{F}(A) \subseteq A$.

Definition: The *forwards orbit* $\mathcal{O}(\mathbf{x})$ is defined as $\{\mathbf{x}, \mathbf{F}(\mathbf{x}), \mathbf{F}^2(\mathbf{x})...\}$.

Backwards orbits are often not well-defined because **F** is not necessarily invertible.

Stability definitions extend as follows:

Definition: An invariant set Λ is *Lyapunov stable* if for any open neighbourhood U of A, there exists an open neighbourhood V of A such that for any $\mathbf{x} \in V$, we have $\mathbf{F}^n(\mathbf{x}) \in U$ for all $n \geq 0$.

Definition: An invariant set Λ is *quasi-asymptotically stable* if there exists an open neighbourhood U of A such that for all $\mathbf{x} \in U$ and for any neighbourhood V of A, there exists an n_0 such that $\mathbf{F}^n(\mathbf{x}) \in V$ for all $n \geq n_0$.

Stability of a fixed point $\mathbf{x}_0$ of $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$ can be determined by considering $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$, which obeys $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{x}_n) - \mathbf{x}_0 \Rightarrow \mathbf{y}_{n+1} = A\mathbf{y}_n + O(|\mathbf{y}|^2)$, where A is the Jacobian matrix:

$$A_{ij} = \frac{\partial F_i}{\partial x_j}.$$

We can show that a fixed point $\mathbf{x}_0$ is:

- asymptotically stable if all eigenvalues of A satisfy $|\lambda| < 1$;
- Lyapunov unstable if there is an eigenvalue of A with $|\lambda| > 1$;
- non-hyperbolic (definition) if there is an eigenvalue of A with $|\lambda| = 1$.

The stability of an N-cycle is determined by repeatedly applying this theory: $\mathbf{y}_1 = A^{(0)}\mathbf{y}_0$, $\mathbf{y}_2 = A^{(1)}\mathbf{y}_1$, ... up until $\mathbf{y}_N = A^{(N-1)}...A^{(0)}\mathbf{y}_0$. The stability of the N-cycle is then determined by the eigenvalues of $A^{(N-1)}...A^{(0)}$.

10.2 Local bifurcations in 1D maps

Bifurcations occur in maps when an eigenvalue cross $|\lambda| = 1$. So in 1D maps, $x_{n+1} = F(x_n, \mu)$, we simply need $F' = \pm 1$ for a bifurcation.

WLOG put fixed point at $x = 0$ and bifurcation point at $\mu = 0$. We can then classify bifurcations as for ODEs.

In general, for the $F' = +1$ case, we have:

$$x_{n+1} = x_n + \nu_0(\mu) + x_n \nu_1(\mu) + \frac{1}{2} x_n^2 \nu_2(\mu) + \dots .$$

When $\nu_0, \nu_1 = O(\mu)$ and $\nu_i = O(1)$ for $i \geq 2$, we get a saddle-node bifurcation. When $\nu_0 = O(\mu^2)$, we get transcritical and pitchfork bifurcations.

(1) **Saddle-node bifurcations:** Normal form is $x_{n+1} = x_n + \mu - x_n^2$.

(2) **Transcritical bifurcations:** Normal form is $x_{n+1} = x_n + x_n(\mu - x_n)$.

(3) **Pitchfork bifurcations:** Normal form is $x_{n+1} = x_n + x_n(\mu \mp x_n^2)$. *Supercritical* if minus sign, *subcritical* for plus sign.

In the $F' = -1$ case, we get a *period-doubling bifurcation*. This can be shown via some gruesome algebra, where the idea is to show that $\mathbf{F}^2$ has a pitchfork bifurcation.

The bifurcation diagram could look like one of two cases, depending on whether there is *supercritical* or *subcritical* period doubling:

10.3 Example: the logistic map

The logistic map equation is $x_{n+1} = \mu x_n(1 - x_n)$. Fixed points are at $x = 0$ and $x = 1 - 1/\mu$.

Checking the stability of the fixed points, at $x = 0$ we have $F' = \mu$ and at $x = 1 - 1/\mu$ we have $F' = 2 - \mu$. Hence there is a transcritical bifrucation at $\mu = 1$, and period-doubling bifurcations at $\mu = -1, 3$.

The bifurcating two cycles satisfy $F^2(x) = x$, which reduces after some algebra (and using $x = 0$ and $x = 1 - 1/\mu$ are roots of this equation) to:

$$\mu^2 x^2 - \mu(\mu + 1)x + \mu + 1 = 0$$

which implies

$$x_{1,2} = \frac{1}{2\mu}\left(\mu + 1 \pm \sqrt{(\mu + 1)(\mu - 3)}\right),$$

so the 2-cycles appear in $\mu > 3$ and $\mu < -1$, so we are in the supercritical case.

The stability is determined by the trick:

$$[F(F(x))]' = F'(x_1)F'(x_2) = \mu^2(1 - 2x_1)(1 - 2x_2)$$

$$= \mu^2\left(1 - \frac{2(\mu + 1)}{\mu} + \frac{4(\mu + 1)}{\mu^2}\right)$$

$$= 4 + 2\mu - \mu^2.$$

So stable if $3 < \mu < 1 + \sqrt{6}$. At $\mu = 1 + \sqrt{6}$, there is a further period-doubling bifurcation. This continues to happen, forever. We get the bifurcation diagram:

11 Chaos

11.1 Devaney's versus Glendinning's chaos

Definition: Let F be a continuous map $F : I \to I$ on a bounded interval $I \subseteq \mathbb{R}$. Let $\Lambda \subseteq I$ be an F-invariant set.

(i) We say that F has *sensitive dependence to initial conditions* on Λ if there exists $\delta > 0$ such that $\forall x \in \Lambda$ and $\forall \epsilon > 0$ there exists $y \in \Lambda$ and some $n > 0$ such that $|x - y| < \epsilon$, but $|F^{(n)}(y) - F^{(n)}(x)| > \delta$.

(ii) We say that F is *topologically transitive* on Λ if for all open sets U, V such that $U \cap \Lambda \neq \emptyset$ and $V \cap \Lambda \neq \emptyset$ there exists n such that $F^{(n)}(U) \cap V \neq \emptyset$. (*Idea:* Apply map enough times to U, get something in V.)

Definition (Devaney's): A map F is *chaotic* on Λ if F is sensitively dependent to initial conditions on Λ, topologically transitive on Λ, and periodic points of F are dense in Λ.

Definition: A map F has a *horseshoe* if there is an open interval $J \subseteq I$ and disjoint sub-intervals $K_0, K_1 \subseteq J$ such that $F(K_0) = F(K_1) = J$ (see diagram).

Definition (Glendinning's): A map F is *chaotic* on Λ if F^n has a horseshoe for some $n \geq 1$.