Part II: Fluid Dynamics II - Revision

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1 Fluids from first principles

1.1 The rate of strain tensor

Consider a fluid with velocity field $\mathbf{u}(\mathbf{x})$, and consider an arbitrary point $\mathbf{x}_0$. Near $\mathbf{x}_0$, we can Taylor expand:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla \mathbf{u}(\mathbf{x}_0) + \dots .$$

Since $\nabla \mathbf{u}$ is a tensor, we can split it into a symmetric and antisymmetric part.

Definition: The symmetric part of $\nabla \mathbf{u}$, namely

$$\frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial u_i}\right),$$

is called the *rate of strain tensor*. We write it as $\mathbf{e} = e_{ij}$. The antisymmetric part is called the *vorticity tensor*, and is written $\boldsymbol{\Omega} = \Omega_{ij}$.

The vorticity tensor corresponds to solid-body rotation, whilst the rate of strain tensor corresponds to stretching of the flow.

Theorem: (i) $\mathbf{e}$ is diagonalisable; (ii) if the fluid is incompressible, then $\mathrm{tr}(\mathbf{e}) = 0$.

Proof: (i) $\mathbf{e}$ is symmetric so diagonalisable. For (ii), note that the trace of $\mathbf{e}$ is

$$\frac{\partial u_i}{\partial x_i} = \nabla \cdot \mathbf{u} = 0,$$

for incompressible fluids. $\square$

From the trace condition, it is clear that if we squeeze an incompressible fluid, it must expand elsewhere (see diagram).

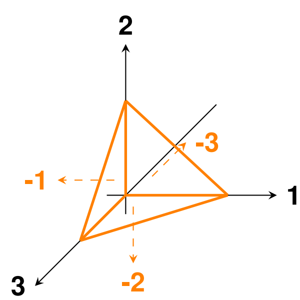
We also note that since $\boldsymbol{\Omega}$ is just a solid-body rotation, only $\mathbf{e}$ contributes to deformation of the flow and hence to *energy loss*.

1.2 The stress tensor

Definition: A *volume force* is a force that acts globally over the whole fluid, e.g. gravity. Forces that act per unit area, e.g. surfaces of one fluid on another, or a fluid on a rigid boundary, are called *surface stresses*. The surface stress exerted on a surface with normal $\hat{\mathbf{n}}$ is written $\boldsymbol{\tau}(\hat{\mathbf{n}})$.

Theorem: $\boldsymbol{\tau}(\hat{\mathbf{n}})$ is linear in $\hat{\mathbf{n}}$.

Proof: Consider an infinitesimal tetrahedron of characteristic size L, with three sides corresponding to coordinate planes, with unit vectors **1**, **2** and **3** say (see diagram).



Let the inclined plane have normal $\hat{\mathbf{n}}$ and area δS. Since we have

$$\frac{\text{volume forces}}{\text{surface forces}} = \frac{O(L^3)}{O(L^2)} \to 0,$$

as $L \to 0$, the surface forces dominate for this scenario.

Projecting areas, we have that the area of the coordinate plane whose normal is $\mathbf{i}$ is $\delta S_i = \hat{n}_i \delta S$. Now balancing forces on the tetrahedron, we have:

$$\mathbf{0} = \boldsymbol{\tau}(\hat{\mathbf{n}})\delta S + \boldsymbol{\tau}(-\mathbf{1})\hat{n}_1\delta S + \boldsymbol{\tau}(-\mathbf{2})\hat{n}_2\delta S + \boldsymbol{\tau}(-\mathbf{3})\hat{n}_3\delta S.$$

By Newton's third law, $\boldsymbol{\tau}(-\mathbf{i}) = -\boldsymbol{\tau}(\mathbf{i})$. Hence it follows that $\boldsymbol{\tau}(\hat{\mathbf{n}})$ is a linear combination of the $\hat{n}_i$. We write $\boldsymbol{\tau}(\hat{\mathbf{n}}) = \boldsymbol{\sigma}\hat{\mathbf{n}}$ where $\boldsymbol{\sigma}$ is the matrix with columns $\boldsymbol{\tau}(\mathbf{i})$. $\square$

Definition: We call $\boldsymbol{\sigma}$ the *stress tensor*. The component σ_{ij} is the ith component of the surface stress on the plane with normal $\mathbf{j}$, pointing into the fluid.

Theorem: The stress tensor $\boldsymbol{\sigma}$ is symmetric.

Proof: Consider angular momentum balance on an arbitrary volume V. Since $\boldsymbol{\sigma}(\hat{\mathbf{n}})$ is the force on the plane with normal $\hat{\mathbf{n}}$ per unit area, the torque per unit area is $\mathbf{x} \times (\boldsymbol{\sigma}\hat{\mathbf{n}})$. We thus have:

$$0 = \iint_{\partial V} \mathbf{x} \times (\boldsymbol{\sigma}\hat{\mathbf{n}})\, dS = \iint_{\partial V} \epsilon_{ijk} x_j \sigma_{kl} n_l \, dS.$$

By the divergence theorem this becomes:

$$0 = \iiint_V \epsilon_{ijk} \frac{\partial}{\partial x_l}(x_j \sigma_{kl})\, dV = \iiint_V \epsilon_{ijk}\sigma_{kj} + \epsilon_{ijk} x_j \frac{\partial \sigma_{kl}}{\partial x_l}\, dV.$$

As the volume shrinks to 0, the x_j term shrinks to zero much faster than the first term. Hence we are left with $\epsilon_{ijk}\sigma_{kj} = 0$, and since V was arbitrary, this implies $\boldsymbol{\sigma}$ is symmetric. □

1.3 Newtonian fluids

Definition: The stress tensor can be decomposed as

$$\sigma_{ij} = -p\delta_{ij} + \sigma_{ij}^{\text{dev}},$$

where $-p\delta_{ij}$ is isotropic, and σ_{ij}^{dev} is traceless. We call σ_{ij}^{dev} the *deviatoric stress* and p the *pressure*.

Definition: A *Newtonian fluid* is such that σ_{ij}^{dev} is linear and instantaneous in $\nabla\mathbf{u}$, i.e.

$$\sigma_{ij}^{\text{dev}} = A_{ijkl} \frac{\partial u_k}{\partial x_l},$$

and the fluid itself is isotropic, i.e. A_{ijkl} is an isotropic tensor.

Theorem: For an incompressible Newtonian fluid, $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$ for a constant μ.

Proof: Recall

$$\sigma_{ij}^{\text{dev}} = A_{ijkl} \frac{\partial u_k}{\partial x_l}.$$

Since A_{ijkl} is isotropic, we must have:

$$\sigma_{ij}^{\text{dev}} = \mu' \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu'' \frac{\partial u_i}{\partial x_j} + \mu''' \frac{\partial u_j}{\partial x_i}.$$

Since incompressible, first term is zero. Since σ_{ij} is symmetric, so is its deviatoric part, which implies $\mu'' = \mu'''$, and we're done. □

Definition: We call μ the *dynamic viscosity*.

1.4 Momentum equations

Theorem (Cauchy's equation): For any fluid (non-Newtonian, compressible) we have:

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{F} + \nabla \cdot \boldsymbol{\sigma}.$$

Proof: Consider sources of change in momentum in an arbitrary volume V. We have:

$$\underbrace{\frac{d}{dt}\left(\int_V \rho u_i \, dV\right)}_{\text{change in momentum}} = \underbrace{-\int_{\partial V} \rho u_i u_j n_j \, dS}_{\text{momentum flux}} + \underbrace{\int_V F_i \, dV}_{\text{body forces}} + \underbrace{\int_{\partial V} \sigma_{ij} n_j \, dS}_{\text{surface stresses}}.$$

Apply the divergence theorem, and use the fact that the volume is arbitrary to deduce:

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) + F_i + \frac{\partial}{\partial x_j}(\sigma_{ij})$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + u_i \underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right)}_{\text{zero, by mass conservation}} + \rho u_j \frac{\partial u_i}{\partial x_j} = F_i + \frac{\partial}{\partial x_j}(\sigma_{ij}),$$

so we're done. □

Theorem (Navier-Stokes' equation): For an incompressible Newtonian fluid, we have:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}.$$

Proof: Substitute $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$ into Cauchy's equation. The result follows by expressing

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and using incompressibility. □

Important note: whenever we use the Navier-Stokes' equations, we must always check the flow is *incompressible*.

1.5 Boundary conditions

Theorem: At a boundary with normal $\hat{\mathbf{n}}$ we require (i) $\mathbf{u}$ continuous; (ii) $\boldsymbol{\sigma}\hat{\mathbf{n}}$ continuous.

Proof: For (i) we prove $\mathbf{u}\cdot\hat{\mathbf{n}}$ and $\mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$ are continuous separately. First we note that

$$0 = \iiint_V \nabla\cdot\mathbf{u}\, dV = \iint_S \mathbf{u}\cdot\hat{\mathbf{n}}\, dS.$$

for a small volume V around the boundary. Shrinking V to zero, so that S lies on the boundary, gives $\mathbf{u}\cdot\hat{\mathbf{n}}^+ = \mathbf{u}\cdot\hat{\mathbf{n}}^-$.

To prove $\mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$ is continuous, suppose there is a discontinuity. Then $|\nabla\mathbf{u}| = \infty$, and so we have infinite stress. Contradiction.

For (ii), let V shrink to zero in the derivation of the Cauchy equation (the first line), taking $F_i = 0$. Since u_i is continuous, the first term vanishes, hence we need $(\rho u_i u_j - \sigma_{ij})n_j$ continuous. But $\mathbf{u}$ is continuous, so we need $\sigma_{ij}n_j$ continuous. □

1.6 Dissipation of energy

Before we prove the main result of this section, we prove a very useful result that will come in handy when studying Stokes' flow.

Theorem: For an incompressible Newtonian fluid, we have

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = 2\mu e_{ij}e_{ij}.$$

Proof: We have:

$$\begin{aligned}
\sigma_{ij}\frac{\partial u_i}{\partial x_j} &= \sigma_{ij}\left(e_{ij}+\Omega_{ij}\right) \\
&= \sigma_{ij}e_{ij} && (\boldsymbol{\Omega}\text{ antisymmetric}) \\
&= -pe_{jj} + 2\mu e_{ij}e_{ij} && (\text{Newtonian fluid }\boldsymbol{\sigma}) \\
&= 2\mu e_{ij}e_{ij}. && (\mathrm{tr}(\mathbf{e})=0)\quad \square
\end{aligned}$$

Theorem: The energy dissipation from internal friction (viscosity) of an incompressible Newtonian fluid is given by

$$2\mu\iiint_V \mathbf{e}:\mathbf{e}\, dV.$$

Proof: We get an energy equation by dotting the Cauchy equation with $\mathbf{u}$. This gives:

$$u_i\left(\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j}\right) = u_iF_i + u_i\frac{\partial \sigma_{ij}}{\partial x_j}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u^2 u_j\right) - \frac{1}{2}\rho u^2\underbrace{\frac{\partial u_j}{\partial x_j}}_{0} =$$

$$\frac{\partial}{\partial x_j}(u_i\sigma_{ij}) - \sigma_{ij}\frac{\partial u_i}{\partial x_j} + u_iF_i.$$

Integrating what's left over a volume V and using the divergence theorem, we have:

$$\underbrace{\frac{d}{dt}\left(\int_V \frac{1}{2}\rho u^2\, dV\right)}_{\text{change in KE}} = \underbrace{-\int_{\partial V}\frac{1}{2}\rho u^2\mathbf{u}\cdot\hat{\mathbf{n}}\, dS}_{\text{KE flux}} + ...$$

$$... + \underbrace{\int_{\partial V}\mathbf{u}\cdot(\boldsymbol{\sigma}\hat{\mathbf{n}})\, dS}_{\substack{\text{rate of work against}\\ \text{surface stresses}}} + \underbrace{\int_V \mathbf{u}\cdot\mathbf{F}\, dV}_{\substack{\text{rate of work against}\\ \text{body forces}}} - \underbrace{\int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j}\, dV}_{\substack{\text{rate of work against}\\ \text{internal stresses}}}.$$

Thus the term we want is:

$$\int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j}\, dV = \int_V 2\mu e_{ij}e_{ij}\, dV,$$

using the earlier Theorem. □

1.7 Scaling analysis

Let U, L, T and P be characteristic speed, length, time and pressure difference of a flow. Define dimensionless variables $\mathbf{u}^* = \mathbf{u}/U$, $\mathbf{x}^* = \mathbf{x}/L$, $t^* = t/T$, $p^* = p/P$. Then the Navier-Stokes' equation becomes:

$$\rho\frac{U}{T}\frac{\partial\mathbf{u}^*}{\partial t^*} + \rho\frac{U^2}{L}\mathbf{u}^*\cdot\nabla^*\mathbf{u}^* = -\frac{P}{L}\nabla^*p^* + \frac{\mu U}{L^2}(\nabla^*)^2\mathbf{u}^*.$$

It turns out that we always need pressure to balance the viscous term. So it follows that:

$$P \sim \frac{\mu U}{L}.$$

Dividing the equation by $\mu U/L^2$, we are left with:

$$\mathrm{Re}\left(\mathrm{St}\frac{\partial\mathbf{u}^*}{\partial t^*} + \mathbf{u}^*\cdot\nabla^*\mathbf{u}^*\right) = -\nabla^*p^* + (\nabla^*)^2\mathbf{u}^*,$$

where we have:

Definition: We define $\nu = \mu/p$ to be the *kinematic viscosity*. We define $\mathrm{Re} = UL/\nu$ to be the *Reynold's number* of the flow. We define $\mathrm{St} = L/(UT)$ to be the *Strouhal number* of the flow.

We can characterise a flow by the values of Re and St. We have:

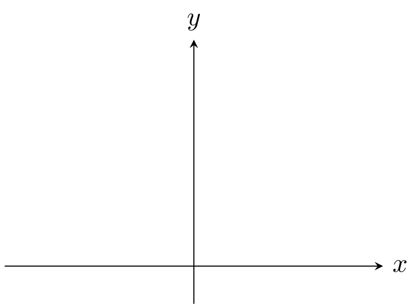
- If $\mathrm{St} \ll 1$, we are in the *quasi-steady regime*. We ignore $\frac{\partial \mathbf{u}^*}{\partial t^*}$.
- If $\mathrm{St} \gg 1$, we are in the *rapidly-oscillating regime*. We ignore $\mathbf{u}^* \cdot \nabla^* \mathbf{u}^*$.
- If $\mathrm{Re} \ll 1$, we are in the *viscously-dominated regime*. We ignore the LHS, i.e. the inertial terms.
- If $\mathrm{Re} \gg 1$, we are in the *inertially-dominated regime*. We ignore the viscosity (though it can be important at boundaries).

Often, $T \sim L/U$, the advection time, giving $\mathrm{St} \approx 1$. So usually only the Reynold's number matters.

1.8 Unidirectional flows

Example: Impulsively started plate

Consider fluid initially at rest in $y > 0$ with a plate at $y = 0$. Suppose the plate suddenly starts moving with velocity U in the x-direction at time $t = 0$.



Seek a solution $\mathbf{u} = (u(y,t), 0, 0)$ to Navier-Stokes. We have the equations

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial p}{\partial x} = 0,$$

subject to the boundary conditions $u \to 0$ as $y \to \infty$, $u = U$ on $y = 0$ and $u \to 0$ as $t \to 0^+$.

Performing a scaling analysis, we see $\delta \sim \sqrt{\nu t}$, where δ is the y length scale. By dimensional analysis then, we have

$$u(y,t) = Uf(\eta),$$

where $\eta = y/\sqrt{\nu t}$. Substituting, we find the ODE for f: $f'' + 2\eta f' = 0$, with boundary conditions $f \to 0$ as $\eta \to \infty$, $f = 1$ on $\eta = 0$ and $f \to 0$ as $\eta \to \infty$.

Hence

$$f(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-s^2}\, ds,$$

from which we have the flow $u(y,t)$.

2 Stokes' flow

2.1 Equations of Stokes' flow

Stokes' flow occurs in the viscously-dominated regime $\mathrm{Re} \ll 1$. The equations of motion are thus:

$$0 = -\nabla p + \mu \nabla^2 \mathbf{u},$$
$$0 = \nabla \cdot \mathbf{u}.$$

where we have absorbed any body forces into a *hydrostatic pressure* p_H defined by $\nabla p_H = \mathbf{F}$.

2.2 Properties of Stokes' flow

(i) Stokes' flow is *instantaneous* because there is no $\partial/\partial t$ term. We thus have $\mathrm{force} \propto \mathrm{velocity}$.

(ii) The equations of Stokes' flow are *linear*. Thus $\mathbf{u}$, p and $\boldsymbol{\sigma}$ are all linear in the applied forces, e.g. $\mathbf{F} = A\mathbf{u}$ by linearity and $\mathrm{force} \propto \mathrm{velocity}$.

(iii) The equations of Stokes' flow are *time-reversible*. This is because $t \mapsto -t$ does nothing to the equations.

(iv) The equations of Stokes' flow are *reversible in space*. This is because letting $\mathbf{x} \mapsto -\mathbf{x}$ and $\mathbf{u} \mapsto -\mathbf{u}$ does nothing to the equations.

(v) All of the functions solving Stokes' flow are harmonic or biharmonic functions.

Taking the divergence of the first equation, we get $\nabla^2 p = 0$ (using incompressibility). Taking the curl of the first equation, we get $\nabla^2 \boldsymbol{\omega} = \mathbf{0}$. Applying the operator $\nabla \times (\nabla \times)$ to the first equation, and recalling that for an incompressible flow, $\nabla \times (\nabla \times) = -\nabla^2$, we get $\nabla^4 \mathbf{u} = \mathbf{0}$.

2.3 Application: sedimentation

Consider a sedimenting cube in a viscous medium. By linearity, we can separate the motion into 3 translations and 3 rotations, each for axes perpendicular the faces of the cube, then superpose at the end.

Consider a rotation about an axis. The force on the cube is linearly related to the angular velocity: $\mathbf{F} = \mathbf{B}\boldsymbol{\Omega}$; since the cube is isotropic, $\mathbf{B} = \lambda \mathbf{I}$ and thus $\mathbf{F} \parallel \boldsymbol{\Omega}$. We now use symmetry to show there is no rotation:

Similarly, we get that drag is parallel to velocity. But drag balances weight less buoyancy, hence the cube sediments straight down.

2.4 Theorems about Stokes' flow

Theorem: The equations of Stokes' flow may be written as:

$$\nabla \cdot \boldsymbol{\sigma} = 0,$$
$$\nabla \cdot \mathbf{u} = 0.$$

Proof: Trivial. □

Theorem (Uniqueness): In a fixed volume V with boundary conditions on ∂V, Stokes' flow is unique up to a solid body rotation.

Proof: Let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two Stokes' flows on V. Define $\mathbf{u} := \mathbf{u}^1 - \mathbf{u}^2$. The idea of the proof is to show $\mathbf{e} = \mathbf{0}$. We have:

$$0 \leq \int_V 2\mu e_{ij} e_{ij} dV = \int_V \sigma_{ij} \frac{\partial u_i}{\partial x_j}\, dV,$$

by the earlier Theorem. Now notice that:

$$\int_V \sigma_{ij} \frac{\partial u_i}{\partial x_j}\, dV = \int_V \frac{\partial}{\partial x_j}(\sigma_{ij} u_i) - u_i \underbrace{\frac{\partial \sigma_{ij}}{\partial x_j}}_{\substack{\text{zero, by} \\ \text{Stokes'} \\ \text{equations}}}\, dV = \int_{\partial V} \sigma_{ij} u_i n_j\, dV,$$

where we use the divergence theorem in the last step. Hence we have:

$$0 \leq \int_V 2\mu e_{ij} e_{ij} dV = \int_{\partial V} \sigma_{ij} u_i n_j\, dS = 0,$$

for a prescribed velocity or stress on the boundary. It follows $\mathbf{e} = \mathbf{0}$. We just need to show this gives solid body rotation. Notice that:

$$\mathbf{0} = \nabla \times \mathbf{e} = \epsilon_{ijk} \frac{\partial}{\partial x_j}(e_{kl}) = \frac{1}{2} \epsilon_{ijk} \frac{\partial^2 u_k}{\partial x_j \partial x_l}.$$

This implies $\nabla \boldsymbol{\omega} = 0$, so vorticity is constant, hence $\boldsymbol{\Omega}$ is constant. Writing out in suffix notation the conditions that $\boldsymbol{\Omega}$ is constant and $\mathbf{e} = \mathbf{0}$, we have:

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0, \qquad \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = 2\Omega_{ij}.$$

Adding gives

$$\frac{\partial u_i}{\partial x_j} = \Omega_{ij} \quad \Rightarrow \quad \nabla \mathbf{u} = \boldsymbol{\Omega}.$$

Integrating this gives

$$\mathbf{u} = \boldsymbol{\Omega} \cdot \mathbf{x} + \mathbf{c},$$

which is just a solid-body rotation. So indeed the flows differ by at most a solid-body rotation. □

Theorem (Minimum dissipation): Let $\mathbf{u}^S$ be a Stokes' flow in a domain V and let $\mathbf{u}$ be an incompressible flow in V, having the same velocity on the boundary ∂V. Then

$$2\mu \int_V \mathbf{e}^S : \mathbf{e}^S\, dV \leq 2\mu \int_V \mathbf{e} : \mathbf{e}\, dV.$$

Proof: Follows in a similar way to uniqueness proof. Consider

$$0 \leq 2\mu \int_V (\mathbf{e} - \mathbf{e}^S) : (\mathbf{e} - \mathbf{e}^S)\, dV.$$

Expanding in a way that mimics the desired result, we see that we want the term

$$4\mu \int_V \mathbf{e}^S : (\mathbf{e}^S - \mathbf{e})\, dV$$

to be zero. Using the exact same tricks as in the uniqueness proof, we can show that indeed this term is zero, and the result follows. □

Theorem (Reciprocal theorem): Let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two Stokes' flows in V with different boundary conditions on ∂V. Then

$$\int_{\partial V} \mathbf{u}^1 \cdot \boldsymbol{\sigma}^2 \hat{\mathbf{n}}\, dS = \int_{\partial V} \mathbf{u}^2 \cdot \boldsymbol{\sigma}^1 \hat{\mathbf{n}}\, dS.$$

Proof: Proceed using the method of the other theorems. Starting with

$$\int_{\partial V} (u_i^1 \sigma_{ij}^2 n_j - u_i^2 \sigma_{ij}^1 n_j)\, dS,$$

apply the divergence theorem and show this is zero by arguments using Stokes' equations, incompressibility, etc. □

2.5 Application of minimum dissipation: geometric bounding

Consider an arbitrary body enclosing a volume V moving at velocity $\mathbf{U}$. Suppose the drag on the body is $\mathbf{D}$. We bound the drag on V as follows.

Upper bound: Let the circumscribed sphere S of the body have radius b. Let $\mathbf{u}^S(\mathbf{x})$ be the Stokes' flow outside the body. Let $\mathbf{u}(\mathbf{x})$ be a flow defined by:

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \text{Stokes' flow outside a sphere, outside the sphere } S, \\ \mathbf{U} \text{ in the gap between } S \text{ and the body.} \end{cases}$$

Then both are valid flows, and both agree on the boundary ∂V. The work done by the drag is

$$-\mathbf{D}^S \cdot \mathbf{U} = 2\mu \int_V \mathbf{e}^S : \mathbf{e}^S \, dV \leq 2\mu \int_V \mathbf{e} : \mathbf{e} \, dV = -\mathbf{D} \cdot \mathbf{U},$$

by the minimum dissipation theorem. But $\mathbf{D} = -6\pi\mu b\mathbf{U}$ (see later) for a sphere of radius b. So we have an upper bound.

Lower bound: Similarly, we can get a lower bound. This time we use an inscribed sphere S' of radius a, with Stokes' flow $\mathbf{u}^S(\mathbf{x})$ outside the sphere S'. We define a flow:

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \text{Stokes' flow outside the body, outside of } V, \\ \mathbf{U} \text{ in the gap between } S' \text{ and the body.} \end{cases}$$

Both are valid flows, but this time the inscribed sphere has the minimum dissipation. So we get a lower bound.

2.6 Application of reciprocal theorem: the resistance matrix

Consider a general rigid body translating with velocity $\mathbf{U}$ and rotating with angular velocity $\mathbf{\Omega}$. By linearity of Stokes' equations, the force and the torque are given by

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{\Omega} \end{pmatrix}.$$

We call the matrix the *resistance matrix*, which depends only on the geometry of the body.

Theorem: $\mathbf{A}$ and $\mathbf{D}$ are symmetric, and $\mathbf{B} = \mathbf{C}^T$.

Proof: Only prove special case $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{G} = \mathbf{0}$. Then we need $\mathbf{F} = \mathbf{AU}$. For two possible $\mathbf{U} = \mathbf{U}^1, \mathbf{U}^2$ then, we get the equation $(A_{ij}U_j^1)U_i^2 = (A_{ij}U_j^2)U_i^1$. Hence interchanging $i \leftrightarrow j$, on the RHS, get result. □

2.7 Corner flows in 2D

We solve 2D corner flows by introducing a *streamfunction*, ψ, obeying

$$\mathbf{u} = \nabla \times (0, 0, \psi).$$

For corner flows, the following properties of the streamfunction are useful:

Theorem: We have:

(i) In plane polars, we have:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}.$$

(ii) The vorticity satisfies $\boldsymbol{\omega} = -\nabla^2\psi\hat{\mathbf{e}}_z$.

(iii) The streamfunction is biharmonic, i.e. $\nabla^4\psi = 0$, in Stokes' flow.

Proof: (i) is trivial. (ii) follows from $\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla \times (\nabla \times (0, 0, \psi)) = -(0, 0, \nabla^2\psi)$, using $\nabla \times (\nabla \times) = -\nabla^2$ for incompressible flow. (iii) follows from $\nabla^2\boldsymbol{\omega} = 0$ for Stokes' flow. □

Example 1: Injection into a corner

Consider a volume flux Q being injected into a corner define by the rays $\theta = -\alpha$ and $\theta = \alpha$ (see diagram).

We seek a steady 2D flow $\mathbf{u} = (u_r, u_\theta, 0)$ with $u_\theta = 0$. Then $u_r \propto Q/r$ by conservation of mass. We thus suppose that the streamfunction has the form $\psi = Qf(\theta)$.

The biharmonic equation is:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{1}{r^2}f''\right) = 0.$$

which reduces to the form $f^{IV} + 4f'' = 0$. Solving gives

$$f(\theta) = A\sin(2\theta) + B\cos(2\theta) + D\theta + C.$$

Symmetry of the flow implies f is odd, so $A = C = 0$. No slip implies $f'(\pm\alpha) = 0$, which gives $2B\cos(2\alpha) + D = 0$. Mass conservation gives

$$Q = \int_{-\alpha}^{\alpha} u_r \, r d\theta = Q(f(\alpha) - f(-\alpha)) \;\Rightarrow\; \frac{1}{2} = B\sin(2\alpha) + D\alpha.$$

Putting all this together, we see

$$f(\theta) = \frac{1}{2}\left(\frac{\sin(2\theta) - 2\theta\cos(2\alpha)}{\sin(2\alpha) - 2\alpha\cos(2\alpha)}\right).$$

Finally, by uniqueness of the solution to Stokes' equations, this must give *the* flow.

Example 2: Scraper flow

Consider a moving scraper inclined at angle $\theta = \alpha$ to a flat plane. Use a frame of reference where the scraper is stationary, and the plane is moving to the left with velocity U (see diagram).

The boundary conditions are $u_r = u_\theta = 0$ on $\theta = \alpha$, and $u_r = -U, u_\theta = 0$ on $\theta = 0$, using no-slip and no-penetration requirements. So the only inhomogeneity is the $-U$ forcing, so we need

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -U,$$

on $\theta = 0$, i.e. trial $\psi \propto r$. In particular, for dimensional consistency, we trial $\psi = Urf(\theta)$ in the biharmonic equation. We obtain:

$$f^{IV} + 2f'' + f = 0,$$

similar to the above example.

2.8 Stokes' flow past a sphere

Consider uniform flow $\mathbf{U}$ past a fixed rigid sphere of radius a centred at the origin.

Theorem: The drag on the sphere is given by $6\pi\mu a\mathbf{U}$.

Proof: We first find the velocity field $\mathbf{u}(\mathbf{x})$ and pressure field $p(\mathbf{x})$ around the sphere.

Linearity of Stokes' equations implies $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$ must be linear in $\mathbf{U}$. Spherical symmetry implies the coefficients must be functions of r alone, and whatever we can build from $\mathbf{x}$ and $\mathbf{U}$:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r),\\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})h(r),\end{aligned}$$

where the μ is included for dimensional consistency.

Now we need only determine f, g and h. Computing

$$\frac{\partial u_i}{\partial x_j}, \quad \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad \frac{\partial p}{\partial x_j},$$

and using these in the incompressibility condition and Stokes' equations gives the governing equations:

$$\begin{aligned}f'/r + 4g + rg' &= 0,\\ f'' + 2f'/r + 2g &= h,\\ g'' + 6g'/r &= h'/r.\end{aligned}$$

Eliminate h then f to get: $r^2 g''' + 11rg'' + 24g' = 0$.

Solving, then back-substituting, we find:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) = \mathbf{U}&\left(-2Ar^2 + B + Cr^{-1} - \frac{1}{3}Dr^{-3}\right)\\ &+ \mathbf{x}(\mathbf{U}\cdot\mathbf{x})(A + Cr^{-3} + Dr^{-5}).\end{aligned}$$

$$p(\mathbf{x}) = \mu(\mathbf{U}\cdot\mathbf{x})(-10A + 2Cr^{-3}).$$

Apply the boundary conditions. We need $\mathbf{u}(\mathbf{x}) = \mathbf{U}$ in the far-field, so $A = 0$, $B = 1$. We need no-slip on the surface of the sphere, so $\mathbf{u} = 0$ on $r = a$, i.e. $C = -\frac{3}{4}a$ and $D = \frac{3}{4}a^3$. Hence the fields reduce to

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right),$$

$$p = -\frac{3a\mu\mathbf{U}\cdot\mathbf{x}}{2r^3}.$$

Now find the stress tensor from $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$, and hence evaluate $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \mathbf{x}/r$ is a unit normal to the sphere. We find that

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\big|_{r=a} = \frac{3\mu}{2a}\mathbf{U} \;\Rightarrow\; \int_{r=a}\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\big|_{r=a}\,dS = 4\pi a^2\frac{3\mu}{2a}\mathbf{U} = 6\pi\mu a\mathbf{U}. \;\square$$

Definition: We can separate the drag on a sphere as:

$$\int_{r=a}(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\hat{\mathbf{n}})\hat{\mathbf{n}}\,dS + \int_{r=a}(\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}\hat{\mathbf{n}})\hat{\mathbf{t}}\,dS,$$

where the first term is the *form drag* and the second is the *skin friction*.

Example: The sedimenting sphere

Consider a sphere of radius a and density ρ_S sedimenting in a fluid of density ρ. How fast does it fall?

We must balance $\text{weight} - \text{buoyancy}$ with the drag. By Archimedes' principle, the upward force exerted on the fluid is equal to the weight of the fluid displaced. Hence:

$$\frac{4}{3}\pi a^3(\rho_S - \rho)\mathbf{g} = 6\pi\mu a\mathbf{U} \quad\Rightarrow\quad \mathbf{U} = \frac{2a^2}{9\mu}(\rho_S - \rho)\mathbf{g}.$$

3 Lubrication theory

3.1 Scaling analysis

Consider a very thin film of fluid of characteristic height h and characteristic length L, with $h \ll L$ (see diagram).

Incompressibility gives:

$$\nabla \cdot \mathbf{u} = 0 \;\Rightarrow\; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \;\Rightarrow\; \frac{u}{L} \sim \frac{v}{h} \;\Rightarrow\; v \sim \frac{hu}{L} \ll u.$$

Hence the flow is essentially uni-directional. From the x-component of Navier-Stokes, we get:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} \\ \Rightarrow \quad \frac{u}{L/u} : \frac{u^2}{L} : \underbrace{\frac{hu}{L}}_{v} \cdot \frac{u}{h} &: \frac{P}{\rho L} : \frac{\nu u}{L^2} : \frac{\nu u}{h^2} \end{aligned}$$

Looking at these various scalings, we have that $u/L^2 \ll u/h^2$ so we can ignore $\nu\frac{\partial^2 u}{\partial x^2}$.

Note that if $u^2/L \ll \nu u/h^2$, then the left hand side vanishes. This condition is equivalent to the *reduced Reynold's number*

$$\mathrm{Re}^* = \frac{uh}{\nu} \cdot \frac{h}{L}$$

being small. On these assumptions, and the inclusion of a body force, the final equation is:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + f_x.$$

By a similar analysis, the y-momentum equation gives:

$$0 = -\frac{\partial p}{\partial y} + f_y.$$

3.2 Conservation of mass

In addition to the above equations, we require conservation of mass. By considering the change in volume in the diagram in a time δt, we get:

$$(q(x) - q(x+\delta x))\delta t = \delta h \delta x \qquad \Rightarrow \qquad \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0.$$

3.3 Equations of lubrication theory

Altogether, we have derived the *2D lubrication equations*:

The 2D Lubrication Equations:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + f_x,$$

$$0 = -\frac{\partial p}{\partial y} + f_y,$$

$$0 = \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x}.$$

By the same derivation, these equations can be generalised to 3D. Let z be the direction in which the surface slowly changes height. Define $\mathbf{u}_2 = (u, v)$, $p = p(x, y, t)$ $\nabla_2 = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and

$$\mathbf{q}_2 = \int_0^h \mathbf{u}_2 \, dz.$$

Then the 3D lubrication equations are:

The 3D Lubrication Equations:

$$0 = -\nabla_2 p + \mu\frac{\partial^2 \mathbf{u}_2}{\partial z^2} + \mathbf{f}_2,$$

$$0 = -\frac{\partial p}{\partial z} + f_z,$$

$$0 = \frac{\partial h}{\partial t} + \nabla_2 \cdot \mathbf{q}_2.$$

In order for these equations to hold, we need the flow to be incompressible, in a thin film $h \ll L$ and the reduced Reynold's number must be small.

3.4 Examples of lubrication theory

Example 1: Thrust bearing

Consider a thrust bearing, as shown.

We identify the geometry first. We have $h(x) = d_1 + \alpha x$, where $\alpha = (d_2 - d_1)/L$. The boundary conditions are $u = 0$ on $y = h(x)$, $u = -U$ on $y = 0$ and $p = p_0$ on $x = L$, $y = d_2$.

We first integrate the second lubrication equation. We have:

$$0 = \frac{\partial p}{\partial y} \quad \Rightarrow \quad p = p(x).$$

This allows us to solve the first lubrication equation. Since p does not depend on y, we can integrate directly:

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \quad \Rightarrow \quad u = -\frac{1}{2\mu}\frac{\partial p}{\partial x} y(h-y) - \frac{U(h-y)}{h}.$$

We can now use the third lubrication equation to find the pressure explicitly. Integrating the expression for u, we get:

$$q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} - \frac{1}{2}Uh.$$

The third lubrication equation gives $q = \text{constant}$ since h does not vary with time. Hence rearranging and integrating, we can get p. Using the pressure boundary condition we can determine q.

Example 2: Cylinder approaching a wall

Consider a cylinder approaching a wall with speed V, as shown.

We identify the geometry first. We have $h - d = a(1 - \cos(\theta)) - Vt$ and $\sin(\theta) = x/a$. Using $\theta \ll 1$, we obtain the approximation:

$$h \approx d\left(1 + \frac{1}{2}\frac{x^2}{ad}\right) - Vt.$$

The boundary conditions are $u = 0$ on $y = 0$, $u = 0$ on $y = h$ and $p \to p_0$ as $|x| \to \infty$.

As usual, start with the second lubrication equation concerning the pressure. We find that $p = p(x)$ since there is no body force. This means we can directly integrate the first lubrication equation giving:

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x} y(h-y).$$

This gives the volume flux:

$$q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}.$$

Applying the third lubrication equation (remember $h_t = -V$, because the cylinder is moving), we find that $q = Vx$. Inserting this into the above equation, we can find the pressure (and fix the constant with the boundary condition for the pressure).

Example 3: Droplet spreading

Consider a 2D droplet of syrup spreading out under gravity (see diagram).

This time, we don't know $h(x)$. Instead, we must find it. We need $u = 0$ on $z = 0$, and continuity of stress implies $p = p_0$ on $z = h$, and

$$\mu \frac{\partial u}{\partial z} = \mu_{\text{air}} \frac{\partial u_{\text{air}}}{\partial z} \quad \Rightarrow \quad \frac{\partial u}{\partial z} \approx 0,$$

on $z = h$, since the viscosity of air is much smaller than the viscosity of the syrup.

We begin, as usual, with the second lubrication equation for the pressure. This time we need a body force:

$$\frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad p = p_0 + \rho g(h - z).$$

So in particular from the first lubrication equation we have:

$$u = -\frac{g}{2\nu}\frac{\partial h}{\partial x} z(2h - z),$$

which gives a volume flux:

$$q = -\frac{gh^3}{3\nu}\frac{\partial h}{\partial x}.$$

Substituting this into the third lubrication equation gives a PDE for h, which we solve using a similarity solution.

The PDE we get is:

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

Let $x \sim L$, $h \sim H$ and $t \sim T$. We want to use dimensional analysis, so have to reduce to just two independent variables and one dimensionless group. To get a relationship between the variables, use global mass conservation:

$$\int_0^{x_n} h(x,t)\, dx = V,$$

which gives the scaling $HL \sim V$. Choosing $T = t$, elapsed time, we get:

$$L \sim \left(\frac{V^3 gt}{\nu}\right)^{1/5}, \quad H \sim \left(\frac{\nu V^2}{gt}\right)^{1/5}.$$

Hence we write:

$$h = \left(\frac{\nu V^2}{gt}\right)^{1/5} f(\eta),$$

where η is the dimensionless variable:

$$\eta = \left(\frac{\nu}{V^3 gt}\right)^{1/5} x.$$

Substituting into the PDE, we obtain:

$$-\frac{1}{5}f - \frac{1}{5}\eta f' = \frac{1}{3}\left(f^3 f'\right)',$$

after some gruesome algebra. This equation can be integrated directly and hence we can find an expression for $h(x)$.

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4 Generation of vorticity

4.1 The vorticity equation

Theorem (The vorticity equation) The Navier-Stokes equations with a conservative body force imply

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

Proof: Take curl of Navier-Stokes equations and use $\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla(\frac{1}{2}\mathbf{u}^2) - (\mathbf{u}\cdot\nabla)\mathbf{u}$. □

We interpret each of the terms in the vorticity equation as follows:

- $(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ is advection of vorticity;
- $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ is vortex stretching;
- $\nu\nabla^2\boldsymbol{\omega}$ is diffusion of vorticity.

4.2 Vortex stretching

Consider two points in a fluid $\mathbf{x}_0$ and $\mathbf{x}_1$, separated by a vector $\boldsymbol{\delta\ell}$ (see the diagram).

In a short time, we have:

$$\mathbf{x}_0(\delta t) = \mathbf{x}_0(0) + \mathbf{u}(\mathbf{x}_0)\delta t,$$
$$\mathbf{x}_1(\delta t) = \mathbf{x}_1(0) + \mathbf{u}(\mathbf{x}_1)\delta t,$$

and so

$$\boldsymbol{\delta\ell}(\delta t) = \boldsymbol{\delta\ell}(0) + [\mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_0)]\delta t$$

$$\Rightarrow \quad \frac{D}{Dt}\boldsymbol{\delta\ell} = \frac{\boldsymbol{\delta\ell}(\delta t) - \boldsymbol{\delta\ell}(0)}{\delta t} = \mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_0).$$

Taylor expand $\mathbf{u}(\mathbf{x}_1)$ about $\mathbf{x}_0$. Then

$$\frac{D}{Dt}\boldsymbol{\delta\ell} = \mathbf{u}(\mathbf{x}_0) + (\mathbf{x}_1 - \mathbf{x}_0)\cdot\nabla\mathbf{u}(\mathbf{x}_0) - \mathbf{u}(\mathbf{x}_0) = (\boldsymbol{\delta\ell}\cdot\nabla)\mathbf{u}(\mathbf{x}_0).$$

Hence *vorticity acts like line elements*. We have: *stretching of fluid elements amplifies vorticity*.

4.3 Diffusion of vorticity

Slogan: Vorticity is generated at rigid boundaries by the no-slip condition, then diffuses away.

This was seen in lectures through many examples, e.g. the impulsively started plate from unidirectional flow.

4.4 Confinement of vorticity: suction flows

Consider flow past a rigid wall of velocity U in the x-direction in the far-field. Suppose there is a suction across the wall that causes a velocity $-V$ in the y-direction (see diagram).

We seek a steady-state solution $\mathbf{u} = (u(y), -V, 0)$ as a solution. Inserting into Navier-Stokes, and using the no-slip condition together with the far-field condition, we get

$$u = U(1 - e^{-Vy/\nu}).$$

There is a steady state when diffusion from the wall, $\delta \sim \sqrt{\nu t}$ (cf impulsively-started plate example), balances advection towards the wall $\delta \sim Vt$, i.e. when $\delta \sim \nu/V$. This is the length scale we see above. Vorticity is confined in a small region near the wall.

5 Boundary layer theory

5.1 Scaling analysis

Consider a boundary layer with interval flow **u** and external, inviscid flow **U** (see the diagram).

Inside the boundary layer, the variables scale as $x \sim L$, $y \sim \delta$, $t \sim L/U$ (the advection time) and $u \sim U$.

Incompressibility gives:

$$\nabla \cdot \mathbf{u} = 0 \;\Rightarrow\; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \;\Rightarrow\; \frac{U}{L} \sim \frac{v}{\delta} \;\Rightarrow v \sim \frac{U\delta}{L} \ll u,$$

if $\delta \ll L$. This is the same as lubrication theory. The Navier-Stokes' equation in the x-direction gives:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{U}{L/U} : \frac{U^2}{L} : \underbrace{\frac{\delta U}{L}}_{v} \cdot \frac{U}{\delta} : \frac{P}{\rho L} : \frac{\nu U}{L^2} : \frac{\nu U}{\delta^2}.$$

Since $\delta \ll L$, we can ignore $\nu\frac{\delta^2 u}{\delta x^2}$. Note that this is still the same as lubrication theory. The difference is that we don't assume that the reduced Reynold's number is small. Instead, requiring that the remaining variables balance gives:

$$\delta \sim \sqrt{\frac{\nu L}{U}}, \qquad P \sim \rho U^2.$$

Navier-Stokes in the y-direction gives:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\delta U/L}{(L/U)} : \frac{U \cdot (\delta U/L)}{L} : \frac{(\delta U/L)^2}{\delta} : \frac{P}{\rho\delta} : \frac{\nu(\delta U/L)}{L^2} : \frac{\nu(\delta U/L)}{\delta^2}.$$

Substituting for ν and P from above, and dividing through by U^2, we see that there is only one appreciable term:

$$0 = \frac{\partial p}{\partial y}.$$

5.2 The boundary layer equations

So far, we have scaled the Navier-Stokes' equations. There is one more step in deriving the boundary layer equations. Let the outer inviscid flow be $\mathbf{U} = (U, 0)$ satisfying:

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) = -\frac{\partial p}{\partial x}.$$

We can substitute for this in the x-direction equation of Navier-Stokes' to get the *boundary layer equations*:

The Boundary Layer Equations:

$$\frac{Du}{Dt} = \frac{DU}{Dt} + \mu\frac{\partial^2 u}{\partial y^2},$$

$$0 = \frac{\partial p}{\partial y},$$

$$0 = \nabla \cdot \mathbf{u},$$

together with the boundary condition $u(x,y) \to U(x)$ as $y \to \infty$; that is, u tends to the outer flow as $y \to \infty$.

5.3 Examples of boundary layer theory

Example 1: The Blasius boundary layer

Consider a semi-infinite flat plate with outer flow constant, $U(x) = U_\infty$ (see diagram). We would like to determine the inner flow.

The only characteristic length scale in the problem is distance from the leading edge of the plate, x. Thus if the boundary layer thickness is δ, we know that

$$\delta \sim \sqrt{\frac{\nu x}{U_\infty}}.$$

Introduce a streamfunction given by $\mathbf{u} = (\psi_y, -\psi_x)$. From the scaling, we must be able to write

$$\psi = U_\infty \delta(x) f(\eta),$$

where η is the dimensionless variable

$$\frac{y}{\delta(x)} = \sqrt{\frac{U_\infty}{\nu x}}y.$$

Substituting everything into the first of the boundary layer equations, we get

$$2f''' + ff'' = 0,$$

after some gruesome algebra.

The no-slip condition on the plate implies that $f' = 0$ when $\eta = 0$, and the no-penetration condition implies that $f = 0$ on $\eta = 0$. The far field condition is that $u \to U_\infty$ as $y \to \infty$, so $f' \to 1$ as $\eta \to \infty$.

Example 2: The 2D momentum jet

Consider a jet not affected by gravity which is steady in time, with no pressure gradient in the x-direction:

In this case, we don't know the outer flow, so we seek a conserved quantity to help. The first of the boundary layer equations is:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2},$$

where we kept pressure instead because we don't know the outer flow. Since there is no pressure gradient in the x-direction, however, the pressure immediately drops out anyway.

Using incompressibility, we can rewrite this equation as:

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = \nu\frac{\partial^2 u}{\partial y^2}.$$

Integrating from $-\infty$ to ∞, and using $u, \frac{\partial u}{\partial y} \to 0$ as $|y| \to \infty$, we have:

$$\frac{d}{dx}\left(\int_{-\infty}^{\infty} u^2 dy\right) = 0 \quad \Rightarrow \quad M = \int_{-\infty}^{\infty} \rho u^2 \, dy$$

is a conserved quantity (the momentum flux). This gives the scaling relationship $\rho U(x)^2\delta(x) \sim M$, where M is known and set by the initial conditions on u. Simultaneously solving with the scaling relationship

$$\delta(x) \sim \sqrt{\frac{\nu x}{U(x)}},$$

we can determine $\delta(x)$ and $U(x)$ in terms of x.

We now follow the normal procedure and introduce a streamfunction **u** $= (\psi_y, -\psi_x)$, with $\psi = U(x)\delta(x)f(\eta)$, and $\eta = y/\delta$. Substituting all of this into the first boundary layer equation gives:

$$3f''' + (f')^2 + ff'' = 0,$$

after some gruesome algebra.

At the jet's nozzle, all fluid must spurt out in the x-direction. So we must have $v = 0$ at $y = 0$, which gives $f(0) = 0$. The far-field condition gives $u \to 0$ as $|y| \to \infty$, so $f' \to 0$ as $|\eta| \to \infty$ (the flow is not disturbed far away). Finally, we need the momentum flux to be constant:

$$\frac{M}{\rho} = \int_{-\infty}^{\infty} u^2 \, dy \quad \Rightarrow \quad \int_{-\infty}^{\infty} (f')^2 \, d\eta = 1.$$

This turns out to be an exactly solvable problem. We can hence find u and v, solving the flow completely.

Using this information, we can prove that the volume flux in the jet scales like $x^{1/3}$; since this increases further downstream, more fluid is being drawn into the jet from the external flow - we call this *entrainment*.

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6 Flow stability

6.1 Kelvin-Helmholtz instabilities

Consider two layers of fluid; suppose the top layer has density ρ_1 and constant velocity U_1 in the x-direction and the bottom layer has density ρ_2 with constant velocity U_2 in the x-direction.

We ignore the viscous boundary layer at the interface, $y = 0$. Perturb the interface to $y = \eta(x,t)$ and consider the limit of small perturbations.

Definition: By *small perturbations*, we mean the amplitude is small with respect to the only length scale in the problem, the *wavelength* of the perturbations, i.e. we require

$$\frac{|\eta|}{\lambda} \ll 1 \quad \Rightarrow \quad |\eta_x| \ll 1.$$

Theorem: If $U_1 \neq U_2$, this flow is unstable to small perturbations (this is called *Kelvin-Helmholtz instability*).

Proof: Let

$$\mathbf{u} = (u, v) = \begin{cases} (U_1, 0) + \nabla\phi_1 \text{ in } y > 0, \\ (U_2, 0) + \nabla\phi_2 \text{ in } y < 0. \end{cases}$$

By incompressibility, we need $\nabla^2\phi_i = 0$ for $i = 1, 2$. This is the equation we must solve (Laplace's equation).

We need some boundary conditions:

(a) The fluid remains undisturbed in the far-field; hence $\phi_1 \to 0$ as $y \to \infty$ and $\phi_2 \to 0$ as $y \to -\infty$.

(b) Kinematic condition: Fluid elements at the interface remain at the interface for all time. Hence we need:

$$\frac{D}{Dt}(y - \eta(x,t)) = 0 \;\Rightarrow\; \cancel{\frac{\partial y}{\partial t}} - \frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla(y - \eta(x,t)) = 0$$

$$\Rightarrow\; -\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + v = 0$$

$$\Rightarrow\; \frac{\partial \phi_i}{\partial y} - \left(U_i + \frac{\partial \phi_i}{\partial x}\right)\frac{\partial \eta}{\partial x} - \frac{\partial \eta}{\partial t} = 0.$$

This holds at $y = \eta(x,t)$.

(c) Dynamic condition: Use the time-dependent Bernoulli theorem. Recall that for potential flow,

$$\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho|\mathbf{u}|^2 + p + \psi = f(t),$$

where ψ is the potential of a conservative force (e.g. gravity), i.e. the LHS does not depend on spatial position. Thus compare at both sides of the boundary, $y = \eta^-$ and $y = \eta^+$:

$$\rho_1\frac{\partial \phi_1}{\partial t} + \frac{1}{2}\rho_1|(U_1, 0) + \nabla\phi_1|^2 + p_1 = f_1(t),$$
$$\rho_2\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\rho_2|(U_2, 0) + \nabla\phi_2|^2 + p_2 = f_2(t).$$

Continuity of pressure implies $p_1 = p_2$, so subtracting, we obtain:

$$\rho_1\frac{\partial \phi_1}{\partial t} + \frac{1}{2}\rho_1|(U_1, 0) + \nabla\phi_1|^2 = \rho_2\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\rho_2|(U_2, 0) + \nabla\phi_2|^2 + f(t),$$

for some space-independent function $f(t)$. This holds at $y = \eta(x,t)$.

We now continue the proof by linearising the boundary conditions. We treat the kinematic and dynamic boundary conditions as if they hold on $y = 0$ instead of $y = \eta(x,t)$. We linearise the kinematic and boundary conditions by removing terms that are quadratic in derivatives (recall $|\eta_x| = O(\epsilon)$, say, and also $|\nabla\phi_i| = O(\epsilon)$). This leaves us with the complete set of linearised equations and boundary conditions:

$$\nabla^2\phi_i = 0,$$

with boundary conditions: $\phi_1 \to 0$ as $y \to \infty$, $\phi_2 \to 0$ as $y \to -\infty$, and on $y = 0$:

$$0 = \frac{\partial \eta}{\partial t} + U_i\frac{\partial \eta}{\partial x} - \frac{\partial \phi_i}{\partial y} = 0,$$

$$\rho_1\frac{\partial \phi_1}{\partial t} + \frac{1}{2}\rho_1\left(U_1^2 + U_1\frac{\partial \phi_1}{\partial x}\right) = \rho_2\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\rho_2\left(U_2^2 + U_2\frac{\partial \phi_2}{\partial x}\right) + f(t),$$

where f is space-independent.

We now set $\eta = A\exp(ikx + \sigma t)$ (i.e. we decompose into Fourier modes), and trial $\phi_i = f_i(y)e^{ikx+\sigma t}$. Laplace's equation then gives:

$$f_i'' - k^2 f_i = 0,$$

and so solving for f_i subject to the far-field boundary conditions, we get $\phi_1 = B_1 e^{-ky}e^{ikx+\sigma t}$ and $\phi_2 = B_2 e^{ky}e^{ikx+\sigma t}$.

Now insert the above trial functions into the kinematic and dynamic boundary conditions. We obtain:

$$\sigma A + ikU_1 A = -kB_1$$
$$\sigma A + ikU_2 A = kB_2$$
$$\sigma B_1 + ikU_1 B_1 = \sigma B_2 + ikU_2 B_2,$$

where $f(t)$ vanishes in the dynamic boundary condition because there is an explicit x-dependence on the LHS.

Substituting for B_1 and B_2 in the third equation, we obtain:

$$-(\sigma + ikU_1)^2 = (\sigma + ikU_2)^2.$$

Hence

$$\sigma = -ik\left(\frac{U_1 + U_2}{2}\right) \pm k\left(\frac{U_1 - U_2}{2}\right),$$

and thus

$$\eta(x,t) = A\exp\left(ik\left(x - \frac{(U_1 + U_2)t}{2}\right)\right)\exp\left(\pm\frac{1}{2}(U_1 - U_2)kt\right)$$

Hence if $U_1 \neq U_2$, there is *always* an unstable mode with positive exponential growth. □

6.2 Mechanism of instability

We have shown that instability occurs, and would like to explain it physically. Between a crest and a trough, vorticity is advected:

The vorticity tends to push the crests up and the troughs down, which results in instability. The fluid rolls up into *Kelvin-Helmholtz billows* and the flow is said to be *turbulent*.

6.3 Stabilisation of Kelvin-Helmholtz

Kelvin-Helmholtz instabilities can be cured by the introduction of:

(i) Gravity: This modifies the dynamic boundary condition, and the resulting equation for σ can have purely imaginary roots if the discriminant is less than zero.

(ii) Surface tension: This modifies the pressure condition to $p_2 - p_1 = -\gamma\eta_{xx}$ across the interface between the fluids, where γ is the *surface tension*. Again, we can get purely imaginary values of σ, given an appropriate discriminant condition.