Part III: General Relativity - Revision

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1 Review of special relativity

1.1 Minkowski spacetime

Definition: *Minkowski spacetime* is the set $\mathbb{R}^4$, together with a metric described by the *line element*:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

The distance between (t, x, y, z) and $(x + dx, y + dy, z + dz, t + dt)$ is ds. More compactly, the line element can be written $ds^2 = \eta_{ab} dx^a dx^b$, where $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

Points are *spacelike-separated* if $ds^2 > 0$, *timelike-separated* if $ds^2 < 0$, and *null-separated* if $ds^2 = 0$.

Definition: The *proper time* of a timelike curve is the time experienced by an observer travelling along that curve. Equivalently, the tangent vector to the curve, u^a, obeys $u^a u^b \eta_{ab} = -1$.

1.2 Symmetries of Minkowski spacetime

The Minkowski metric is clearly invariant under translations. So restrict attention to symmetries which fix the origin.

Theorem: Let $\mathbf{x} \mapsto \mathbf{x}' = \Lambda \mathbf{x}$ be a symmetry of Minkowski spacetime fixing the origin. Then $\Lambda^T \eta \Lambda = \eta$.

Proof: Since the transformation is a symmetry, it must map straight lines to straight lines. So Λ is a linear map. We need the line element to be invariant, hence:

$$ds'^2 = dx'^T \eta dx' = dx^T \Lambda^T \eta \Lambda dx = dx^T \eta dx = ds^2.$$

Holds for all dx, so result follows. □

Note this generalises rotations, for which we have: $R^T \mathbb{I}_3 R = \mathbb{I}_3$.

Definition: Any Λ obeying $\Lambda^T \eta \Lambda = \eta$ is called a *Lorentz transformation*.

Theorem: Lorentz transformations Λ form a group.

Proof: Note $\det(\Lambda^T \eta \Lambda) = \det(\eta) \Rightarrow \det(\Lambda)^2 = 1$. So always invertible. Axioms are then trivial to check. □

Definition: The group of Lorentz transformations is called the *Lorentz group*, written $O(1,3)$. If we add the translations back in, the group is called the *Poincaré group*.

Example: A familiar example of a Lorentz transformation is the *Lorentz boost*. For example, for a boost by velocity v in the x-direction, the formula is:

$$t' = \frac{t - vx}{\sqrt{1 - v^2}}, \quad x' = \frac{x - vt}{\sqrt{1 - v^2}}, \quad y' = y, \quad z' = z.$$

To get a general Lorentz boost, first rotate coordinates, apply the boost in the x-direction, then rotate back, viz $R^T \Lambda_x R$.

1.3 Indexed objects

Definition: An object with a single upstairs index, v^a, is a *contravariant vector*.

Definition: Given a contravariant vector v^a, we can define a *covariant vector* via: $v_a = \eta_{ab} v^b$.

Conversely, given a covariant vector v_a we can use the inverse metric to recover the contravariant vector: $v^a = \eta^{ab} v_b$. Remember the slogan:

The metric can be used to lower indices. The inverse metric can be used to raise indices.

1.4 Lorentz transformations and indices

Clearly we want contravariant vectors to transform in the natural way: $v^a \mapsto \Lambda^a{}_b v_b$.

Definition: A *scalar quantity* is invariant under Lorentz transformations.

We want inner products to be scalars, so we have:

Theorem: Under a Lorentz transformation, covariant vectors transform as $\omega_a \mapsto \Lambda^b{}_a \omega_b$.

Proof: Consider scalar $\phi = \omega_a v^a$, and insert transformation law for vector. Use scalars invariant under Lorentz transformations. □

Definition: A *tensor* of type (r, s) in special relativity is an object $T^{a_1 a_2 \dots a_r}{}_{b_1 b_2 \dots b_r}$ which transforms under a Lorentz transformation as:

$$T^{a_1' a_2' \dots a_r'}{}_{b_1' b_2' \dots b_r'}$$
$$= \Lambda^{a_1'}{}_{a_1} \Lambda^{a_2'}{}_{a_2} \dots \Lambda^{a_r'}{}_{a_r} T^{a_1 a_2 \dots a_r}{}_{b_1 b_2 \dots b_r} \Lambda^{b_1}{}_{b_1'} \Lambda^{b_2}{}_{b_2'} \dots \Lambda^{b_r}{}_{b_r'}$$

1.5 Maxwell's equations

Definition: Denote derivatives in special relativity by

$$\partial_a = \frac{\partial}{\partial x^a}.$$

Definition: The *electromagnetic field strength tensor* is:

$$F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

The *four current* is $j^a = (\rho, \mathbf{j})$, where **J** is the 3-current density and ρ is the charge density.

Theorem: Maxwell's equations may be written:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0, \quad \partial_a F^{ab} = -j^b.$$

Proof: Expand out and check. □

2 Differential geometry

2.1 Equivalence principles

Principle 1: Inertial and gravitational mass are the same. Inertial mass appears in Newton's law: $F = m_{\text{inertial}} a$. Gravitational mass appears in the law of gravitation:

$$F = \frac{G m_{\text{grav 1}} m_{\text{grav 2}}}{r^2}.$$

Principle 2: In a freely-falling frame, the laws of physics are those of special relativity.

2.2 Manifolds and vectors

Definition: A *manifold* M is a space we can label with coordinates. Let the coordinates be x^a in general.

Definition: A *vector* at a point p on a manifold is a differential operator of the form

$$V = V^a \frac{\partial}{\partial x^a}\bigg|_p.$$

Vectors give derivatives of functions along curves. Let f be a function and $x^a(t)$ be a curve. The derivative of f along $x^a(t)$ at p is:

$$\frac{df}{dt}\bigg|_p = \frac{\partial x^a}{\partial t} \frac{\partial f}{\partial x^a}\bigg|_p.$$

This is indeed a vector acting on the function f.

Note also that vectors as we've defined them naturally form a *vector space* at any point p.

Definition: The vector space of vectors at p is denoted $T_p(M)$ and is called the *tangent space*. We can choose a basis for this space to be

$$\left\{ \frac{\partial}{\partial x^a} \right\}.$$

This basis is called a *coordinate basis*.

Theorem: Under a bijective, differentiable transformation $\tilde{x}^{a'} = \tilde{x}^{a'}(x^b)$, the components of a vector $V = V^a \partial_a$ transform as:

$$V^a \mapsto \frac{\partial \tilde{x}^{a'}}{\partial x^a} V^a.$$

Proof: The vector V is coordinate independent, so does not change. However, the coordinate basis vectors do, using the chain rule. The result follows immediately. □

2.3 One-forms

Definition: An element of $T_p^*(M)$, the dual space, is called a *covector* or *one-form*.

We can define a natural inner product between one-forms and vectors as follows. Let $V = V^a E_a$ be the expansion of a vector in a (not necessarily coordinate) basis. Let $\{E^a\}$ be the dual basis of one-forms, then mirror the expansion by writing one-forms as $\omega = \omega_a E^a$. The inner product is then clearly defined as:

$$\langle \omega, V \rangle = \omega_a V^b \langle E^a, E_b \rangle = \omega_a V^a,$$

using $\langle E^a, E_b \rangle = \delta^a{}_b$, as is natural for a dual basis.

Theorem: Under a bijective, differentiable transformation $\tilde{x}^{a'} = \tilde{x}^{a'}(x^b)$, the components of a one-form (written in the dual basis to a coordinate basis of vectors) transform as:

$$\omega_a \mapsto \frac{\partial \tilde{x}^a}{\partial x^{a'}} \omega_a.$$

Proof: Same as for vectors. Use invariance of inner product $\langle \omega, V \rangle = \omega_a V^a$ and transformation law for a vector. □

2.4 Differentials

Definition: The *differential* of the function f at the point p is the one-form df at p obeying:

$$\langle df, X\rangle = X(f),$$

for all vectors X at p.

Theorem: $\{dx^a\}$ is the dual basis to $\{\partial_a\}$.

Proof: We have

$$\langle dx^a, \partial_b\rangle = \frac{\partial x^a}{\partial x^b} = \delta^a{}_b. \quad \square$$

We can interpret the differential geometrically. If $\langle df, X\rangle = 0$ in n dimensions, this gives 1 equation, so X has $n-1$ degrees of freedom. So the equation defines an $n-1$ dimensional surface.

On the surface, $\langle df, X\rangle = X(f) = 0$, so $f =$ constant on the surface. Finally, $\langle df, X\rangle = 0$ implies df is orthogonal to tangents on the surface. So df is the normal to the $n-1$ dimensional surface $f =$ constant.

2.5 Tensors

Definition: Let $\{E^a\}$ be a basis of one forms and $\{E_a\}$ be a basis of vectors. A type (r, s) *tensor* is an object:

$$T = T^{a_1\ldots a_r}{}_{b_1\ldots b_s} E_{a_1} \otimes \ldots \otimes E_{a_r} \otimes E^{b_1} \otimes \ldots \otimes E^{b_s}.$$

This immediately gives the transformation law for tensors:

$$T^{a_1\ldots a_r}{}_{b_1\ldots b_s} \mapsto T^{a'_1\ldots a'_r}{}_{b'_1\ldots b'_s} \frac{\partial \tilde{x}^{a'_1}}{\partial x^{a_1}} \cdots \frac{\partial \tilde{x}^{a'_r}}{\partial x^{a_1}} \frac{\partial x^{b_1}}{\partial \tilde{x}^{b'_1}} \cdots \frac{\partial x^{b_s}}{\partial \tilde{x}^{b'_s}}.$$

Definition: Let $X_{a_1\ldots a_r}$ be the components of a tensor. Then the *symmetrisation* of X has components:

$$X_{(a_1\ldots a_r)} = \frac{1}{r!}\sum_{\sigma\in S_r} X_{\sigma(a_1\ldots a_r)}.$$

The *antisymmetrisation* of X has components:

$$X_{[a_1\ldots a_r]} = \frac{1}{r!}\sum_{\sigma\in S_r} \epsilon(\sigma) X_{\sigma(a_1\ldots a_r)},$$

where ϵ is the sign of the permutation.

2.6 p-forms and the wedge product

Definition: An antisymmetric $(0, p)$ tensor is called a p-*form*. If $A_{a_1\ldots a_p}$ are the components of a p-form, antisymmetry means that we have: $A_{a_1\ldots a_p} = A_{[a_1\ldots a_p]}$.

There is a natural multiplication of p-forms through the *wedge product*:

Definition: The *wedge product* of the p-form $A_{a_1\ldots a_p}$ and the q-form $B_{b_1\ldots b_q}$ is defined to be the $(p+q)$-form with components:

$$(A\wedge B)_{a_1\ldots a_p b_1\ldots b_q} = \frac{(p+q)!}{p!q!} A_{[a_1\ldots a_p} B_{b_1\ldots b_q]}.$$

Hence we can write a normal p-form as the sum over the products of basis one-forms:

$$A = \frac{1}{p!} A_{a_1\ldots a_p} E^{a_1} \wedge \ldots \wedge E^{a_p}.$$

Theorem: The wedge product obeys (i) $A\wedge B = (-1)^{pq} B\wedge A$; (ii) $A\wedge A = 0$ if p is odd.

Proof: (ii) follows immediately from (i). For (i), notice for any two coordinates x^a, x^b:

$$(dx^a \wedge dx^b)_{cd} = (dx^a)_c (dx^b)_d - (dx^a)_d (dx^b)_c = -(dx^b \wedge dx^a)_{cd}.$$

So just write $A\wedge B$ out in components, and drag the B basis one-forms to the left of the A ones. $\square$

Definition: The *exterior derivative* of the p-form A is the $(p+1)$-form with components:

$$(dA)_{ba_1\ldots a_p} = (p+1)\frac{\partial}{\partial x^{[b}} A_{a_1\ldots a_p]}.$$

Note: We can rewrite this as:

$$(dA)_{ba_1\ldots a_p} = \frac{1}{p!}\partial_b A_{a_1\ldots a_p},$$

because the components of a p-form are totally antisymmetric.

Theorem: The exterior derivative obeys (i) $d(A\wedge B) = dA\wedge B + (-1)^p A\wedge dB$ (A is a p-form and B is a q-form); (ii) $d(dA) = 0$.

Proof: (i) We have:

$$d(A\wedge B) = \frac{1}{p!q!}\partial_i (A_{i_1\ldots i_p} B_{j_1\ldots j_q}) dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{j_q}$$

$$= \frac{1}{p!q!}\left((\partial_i A_{i_1\ldots i_p}) B_{j_1\ldots j_q} + A_{i_1\ldots i_p} \partial_i B_{j_1\ldots j_q}\right) dx^i \wedge \ldots \wedge dx^{j_q}$$

$$= dA\wedge B + \frac{1}{p!q!} A_{i_1\ldots i_p} (\partial_i B_{j_1\ldots j_q})(-1)^p dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^i \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}$$

$$= dA\wedge B + (-1)^p A\wedge dB.$$

For (ii), simply note that

$$d(dA) \propto (\partial_i \partial_j (A_{\ldots})) dx^i \wedge dx^j \wedge \ldots,$$

which is symmetric on i, j in the derivatives, and antisymmetric in the wedge product, so must be zero. $\square$

2.7 The metric tensor

In GR, we upgrade the line element to $ds^2 = g_{ab}(x^c)dx^a dx^b$, where now g is a symmetric tensor, which is a function of spacetime x^c.

The fact that g is symmetric naturally encodes the equivalence principle: symmetric tensors are diagonalisable, hence there exists a coordinate transformation with:

$$\eta_{a'b'} = \frac{\partial x^a}{\partial \tilde{x}^{a'}} \frac{\partial x^b}{\partial \tilde{x}^{b'}} g_{ab}.$$

The signs are correct because Sylvester's Law of Inertia ensures signature is preserved under change of basis.

Same as Minkowski space, we can raise and lower indices using the metric and its inverse:

$$V_a = g_{ab}V^b, \quad V^a = g^{ab}V_b.$$

2.8 The covariant derivative

Idea: Take an (r, s) tensor, and make an $(r, s+1)$ tensor via a 'derivative' operation.

Scalar: For a scalar ϕ, just use $\partial_a \phi$. It's easy to check this transforms as a vector.

Vector: Try $V^b \mapsto \partial_b V^a$. However, under a coordinate transform:

$$\partial_{b'} \tilde{V}^{a'} = \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial}{\partial x^b} \left(\frac{\partial \tilde{x}^{a'}}{\partial x^a} V^a \right)$$
$$= \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial \tilde{x}^{a'}}{\partial x^a} \partial_b V^a + \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial^2 \tilde{x}^{a'}}{\partial x^b \partial x^a} V^a.$$

So transforms as a tensor plus a non-tensorial term. This suggests we should define:

Definition: The *covariant derivative* of a vector V^a is written $\nabla_b V^a$, and is defined by:

$$\nabla_b V^a = \partial_b V^a + \Gamma^a_{bc} V^c,$$

where Γ^a_{bc} is a correction ensuring $\nabla_b V^a$ transforms as a tensor. We call Γ the *connection*.

Theorem: For the covariant derivative to be tensorial, the connection must transform as:

$$\tilde{\Gamma}^{a'}_{b'c'} = \frac{\partial \tilde{x}^{a'}}{\partial x^a} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}} \Gamma^a_{bc} - \frac{\partial^2 \tilde{x}^{a'}}{\partial x^b \partial x^c} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}}.$$

Proof: Clear from transformation law for $\partial_b V^a$. □

Theorem: For a covariant derivative ∇ which (i) is a linear operation; (ii) obeys the Leibniz rule; we have:

$$\nabla_b V_a = \partial_b V_a - \Gamma^c_{ba} V_c,$$

$$\nabla_c (T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}) = \partial_c T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} + \Gamma^{a_1}{}_{cd} T^{d a_2 \ldots a_r}{}_{b_1 \ldots b_s} + \ldots$$
$$+ \Gamma^{a_r}{}_{cd} T^{a_1 \ldots d}{}_{b_1 \ldots b_s} - \Gamma^d{}_{cb_1} T^{a_1 \ldots a_r}{}_{d b_2 \ldots b_s} - \ldots - \Gamma^d{}_{cb_s} T^{a_1 \ldots a_r}{}_{b_1 \ldots d}.$$

Proof: Assuming the covariant derivative formula for a one-form V_a, the tensor formula follows by the Leibniz rule.

To get the one-form formula. Let $S = W^a V_a$. Then:

$$\partial_b S = \nabla_b S = (\nabla_b W^a) V_a + (\nabla_b V_a) W^a$$
$$= (\partial_b W^a) V_a + \Gamma^c_{ba} W^a V_c + (\nabla_b V_a) W^a.$$

Also, $\partial_b S = (\partial_b W^a) V_a + (\partial_b V_a) W^a$. The result follows. □

2.9 Torsion

Theorem: The difference $\Gamma^a_{bc} - \Gamma^a_{cb}$ transforms as a tensor.

Proof: Trivial from above transformation property. □

Definition: The *torsion tensor* is defined by:

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}.$$

Theorem: $[\nabla_a, \nabla_b] S = T^c_{ab} \nabla_c S.$

Proof: Just write everything out. □

This shows that the torsion tensor measures how much covariant derivatives fail to commute when acting on a scalar. We will assume torsion is zero in this course.

2.10 The metric connection

Definition: The *metric connection* is a torsion-free connection such that $\nabla_c g_{ab} = 0$.

Theorem: With the metric connection, the connection is given by:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc} \right).$$

Proof: Write out $0 = \nabla_a g_{bc}$, $0 = \nabla_b g_{ca}$ and $0 = \nabla_c g_{ab}$. Add the first two equations, and subtract the third. Use symmetry of connection on downstairs indices to finish. □

What's the point of the metric connection? It allows us to raise and lower indices *through* a covariant derivative: $\nabla_b V_a = \cancel{(\nabla_b g_{ac})} V^c + g_{ac} \nabla_b V^c = g_{ac} \nabla_b V^c$.

To lower indices through a covariant derivative, we need:

Theorem: With the metric connection, $\nabla_c g^{ab} = 0$.

Proof: Note that

$$\nabla_l(\delta^i{}_j) = \nabla_l(\delta^i{}_k\delta^k{}_j) = \delta^i{}_k\nabla_l(\delta^k{}_j) + \delta^k{}_j\nabla_l(\delta^i{}_k) = 2\nabla_l(\delta^i{}_j).$$

Hence $\nabla_c(\delta^a{}_b) = 0$. So take covariant derivative of $\delta^a{}_b = g^{ad}g_{db}$ to get result. □

2.11 The Riemann tensor

We saw that for a torsion-free connection, covariant derivatives commute on a scalar. What about on a one-form?

Definition: The *Riemann tensor* $R_{abc}{}^d$ is defined through:

$$[\nabla_a, \nabla_b]V_c = R_{abc}{}^d V_d.$$

Since the LHS is a tensor, and V_d is a tensor, the Riemann tensor is also a tensor.

Theorem: In terms of the connection,

$$R_{abc}{}^d = -\partial_a\Gamma^d_{bc} + \partial_b\Gamma^d_{ac} - \Gamma^e_{bc}\Gamma^d_{ea} + \Gamma^e_{ac}\Gamma^d_{eb}.$$

Proof: Via a short calculation. □

Theorem (Ricci identity): For an arbitrary tensor:

$$[\nabla_e, \nabla_f]T^{ab\ldots}{}_{cd\ldots} = R_{ef}{}^a{}_p T^{pb\ldots}{}_{cd\ldots} + R_{ef}{}^b{}_p T^{ap\ldots}{}_{cd\ldots} + \ldots$$
$$+ R_{efc}{}^p T^{ab\ldots}{}_{pd\ldots} + \ldots$$

Proof: Simple consequence of linearity and Leibniz rule of covariant derivative. □

2.12 Symmetries of the Riemann tensor

Theorem: The Riemann tensor possesses the following symmetries:

1. $R_{abcd} = -R_{bacd}$.
2. $R_{abcd} = -R_{abdc}$.
3. $R_{abcd} = R_{cdab}$.
4. $R_{abcd} + R_{acdb} + R_{adbc} = 0$. This is called the *first Bianchi identity*.

Proof: See later when we discuss normal coordinates. Though notice first is obvious. □

Theorem: As a consequence of these symmetries, the Riemann tensor has:

$$\frac{1}{12}d^2(d^2-1)$$

independent components in d dimensions.

Proof: Consider the possible indices on R_{abcd}. If all the indices are the same, $R_{aaaa} = 0$, by antisymmetry.

By antisymmetry on first and last two indices, only non-vanishing components with 2 distinct indices are: R_{abab}, R_{abba}, R_{baba}, R_{baab}. But these obey $R_{abab} = -R_{abba} = -R_{baba} = R_{baab}$, hence only one independent component. There are $\binom{d}{2}$ ways of picking the two, so that many components.

For three indices, the only non-vanishing components are $R_{abac} = -R_{abca} = R_{baca} = -R_{baac} = R_{acab} = -R_{caab} = -R_{acba} = R_{caba}$, so only one independent component. There are $d\binom{d-1}{2}$ ways of picking the indices.

For four distinct indices, there are $d(d-1)(d-2)(d-3)$ arrangements of $abcd$. This overcounts by $2\times2\times2 = 8$ (antisymmetry on first two, last two, and swap symmetry), and since $R_{abcd} + R_{acdb} = -R_{adbc}$ it overcounts by a further factor of $3/2$. Hence total:

$$\binom{d}{2} + d\binom{d-1}{2} + \frac{d(d-1)(d-2)(d-3)}{12} = \frac{1}{12}d^2(d^2-1). \quad \square$$

2.13 Tensors related to the Riemann tensor

Definition: The *Ricci tensor* is defined by $R_{bd} = R_{abcd}g^{ac}$ (i.e. contracting the first and third indices of the Riemann tensor).

By the symmetries of the Riemann tensor, $R_{ab} = R_{ba}$, i.e. the Ricci tensor is a symmetric tensor.

Definition: The *Ricci scalar* is defined by $R = R_{ab}g^{ab}$.

3 Geodesics

3.1 The geodesic equation

Consider curves on a manifold. The length of a curve $x^a(\lambda)$ between points p and q is:

$$S = \int_p^q ds = \int_p^q \sqrt{\left| g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right|} d\lambda.$$

Definition: A curve that extremises the distance functional is called a *geodesic*.

Theorem: Geodesics $x^a(\lambda)$ obey the geodesic equation:

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0.$$

Proof: Choose to extremise $I = \int_p^q ds^2$. Doing so, get the result. At one point, we need to split a term and swap dummy indices. □

Note: Introducing the tangent vector $V^a = \frac{dx^a}{d\lambda}$, we can rewrite the geodesic equation as:

$$V^b \nabla_b V^a = 0.$$

3.2 Examples

Example 1: Consider the metric:

$$ds^2 = -t^{-2} dt^2 + t^{-2} dx^2.$$

The associated Lagrangian is

$$\mathcal{L} = -t^{-2} \dot{t}^2 + t^{-2} \dot{x}^2,$$

with $\dot{} = d/d\lambda$. This has no explicit x dependence, so we get a first integral $2t^{-2}\dot{x} =$ constant $\Rightarrow \dot{x} = Ct^2$.

There is also no explicit dependence on the parameter λ. Hence the quantity

$$\mathcal{L} - \dot{x} \frac{\partial \mathcal{L}}{\partial x} - \dot{t} \frac{\partial \mathcal{L}}{\partial t}$$

is constant. It follows that $\dot{t}^2 - \dot{x}^2 = Kt^2$ for K a constant.

Eliminate $\dot{x}$ from our conserved quantities to get an equation for $\dot{t}$: $\dot{t}^2 - C^2 t^4 = Kt^2$. Change variable from λ to x via:

$$\dot{t} = \frac{dx}{d\lambda} \frac{dt}{dx} = \dot{x} \frac{dt}{dx} = Ct^2 \frac{dt}{dx}.$$

Then the equation becomes:

$$C^2 t^4 \left(\frac{dt}{dx} \right)^2 - C^2 t^4 = Kt^2.$$

This is now easily integrated to find that the geodesics take the form $(x - x_0)^2 - t^2 = A$, for constant x_0 and A, i.e. they are hyperbolae.

Example 2: Consider the action for a charged particle travelling in spacetime:

$$I = \int ds \left(-m \sqrt{-g_{ab} \dot{x}^a \dot{x}^b} + q A_a \dot{x}^a \right).$$

This is a generalisation of the distance functional we saw above. Here, q is the charge of the particle, m is the mass and A_a is the four-potential of the electromagnetic field. The field strength tensor is $F_{ab} = \partial_a A_b - \partial_b A_a$.

Suppose we work with proper time τ. Then we can set $\sqrt{-g_{ab}\dot{x}^a \dot{x}^b} = 1$ along the geodesic. Computing the relevant derivatives for the Euler-Lagrange equations, we find:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^a} = m \dot{x}^b g_{ab} + q A_a.$$

$$\frac{\partial \mathcal{L}}{\partial x^a} = \frac{1}{2} m (\partial_a g_{cd}) \dot{x}^c \dot{x}^d + q (\partial_a A_b) \dot{x}^b$$

Thus the Euler-Lagrange equation is:

$$0 = m \ddot{x}^b g_{ab} + m \dot{x}^b \dot{x}^c \partial_c g_{ab} + q \dot{x}^b \partial_b A_a - \frac{1}{2} m \dot{x}^c \dot{x}^d \partial_a g_{cd} - q \dot{x}^b \partial_a A_b$$
$$\Rightarrow q F^a{}_b \dot{x}^b = m \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c.$$

3.3 Affine parametrisation

Theorem: Under a transformation of the parameter $\lambda \mapsto \tilde{\lambda}(\lambda)$, the geodesic equation transforms to:

$$\frac{d^2 x^a}{d\tilde{\lambda}^2} + \Gamma^a_{bc} \frac{dx^b}{d\tilde{\lambda}} \frac{dx^c}{d\tilde{\lambda}} = f(\lambda) \frac{dx^a}{d\tilde{\lambda}},$$

where

$$f(\lambda) = -\frac{(d^2\tilde{\lambda}/d\lambda^2)}{(d\tilde{\lambda}/d\lambda)}.$$

Proof: Simple calculation. □

In terms of the tangent vector, $\tilde{V}^a = dx^a/d\tilde{\lambda}$, the geodesic equation becomes:

$$\tilde{V}^b \nabla_b \tilde{V}^a = f(\lambda) \tilde{V}^a.$$

Both of these are the geodesic equation! But in a more general form.

Definition: If $f(\lambda) = 0$, we say the geodesic is *affinely parametrised*. Clearly, a geodesic remains affinely parametrised under a transformation iff the transformation is such that $\tilde{\lambda} = a\lambda + b$ for some constants a and b.

Theorem: For an affinely parametrised geodesic, the length of the tangent vector is preserved along the geodesic.

Proof: $V^a \nabla_a (V^b V_b) = 2 V_b V^a \nabla_a V^b = 0.$ □

3.4 Normal coordinates

Theorem: The coordinate transform:

$$\tilde{x}^a = (x^a - x^a_0) + \frac{1}{2}\Gamma^a_{bc}(x^b - x^b_0)(x^c - x^c_0) + ...$$

sends the connection to zero at x_0.

Proof: Compute:

$$\frac{\partial \tilde{x}^a}{\partial x^e} = \delta^a{}_e + \Gamma^a_{bc}(x^b - x^b_0)\delta^c{}_e + ...$$
$$\frac{\partial^2 \tilde{x}^a}{\partial x^e \partial x^f} = \Gamma^a_{ef} + ...$$

Insert into transformation law for connection to see that in these coordinates it is zero. □

In such coordinates, the derivatives of the metric must vanish, since $0 = \nabla_a g_{bc} = \partial_a g_{bc} - 2\Gamma^d_{ab} g_{dc} = \partial_a g_{bc}$ in these coordinates. Hence the metric is of the form:

$$g_{ab} = c_{ab} + O((x - x_0)^2).$$

A change of basis then sends $c_{ab} \mapsto \eta_{ab}$.

Definition: Coordinates in which the metric takes the above form are called *normal* or *inertial coordinates*.

3.5 Applications of normal coordinates

Theorem: The symmetries of the Riemann tensor hold.

Proof: In normal coordinates, $R_{abcd} = g_{be}\left(\partial_d \Gamma^e_{ca} - \partial_c \Gamma^e_{da}\right)$. Expand the connection in terms of the metric to find:

$$R_{abcd} = \frac{1}{2}\left(\partial_a \partial_d g_{bc} + \partial_c \partial_b g_{ad} - \partial_b \partial_d g_{ac} - \partial_a \partial_c g_{bd}\right).$$

From here, the standard symmetries are easily verified. □

Theorem: In normal coordinates at the point x_0, the metric takes the form:

$$g_{ab} = \eta_{ab} - \frac{1}{3}R_{acbd}(x^c - x^c_0)(x^d - x^d_0) + O((x - x_0)^3).$$

Proof: We know the first two terms in normal coordinates. For the second order term, we need to compute:

$$\partial_c \partial_d g_{ab} = \partial_c(\cancel{\nabla_d g_{ab}} + \Gamma^e_{ad} g_{eb} + \Gamma^e_{bd} g_{ae}) = g_{eb}\partial_c \Gamma^e_{da} + g_{ea}\partial_c \Gamma^e_{db},$$

using $\partial_c g_{ab} = 0$ in normal coordinates.

To make further progress, need an identity for the connection. In normal coordinates, the geodesic equation is $\ddot{x}^a = 0$, so geodesics take the form $x^a(s) = x^a_0 + s\xi^a$, for some constant vector $\xi^a = \dot{x}^a(0)$.

Near x_0, the equation is: $\dfrac{d^2 x^a}{ds^2} + \Gamma^a_{bc}\dfrac{dx^b}{ds}\dfrac{dx^c}{ds} = 0.$

Take derivative wrt s and impose normal coordinates to get, at x_0:

$$\partial_d \Gamma^a_{bc}\xi^b \xi^c \xi^d = 0.$$

Since ξ are arbitrary, have $\partial_d \Gamma^a_{bc} + \partial_b \Gamma^a_{cd} + \partial_c \Gamma^a_{db} = 0$. Using this identity, we have

$$\begin{aligned} 3\partial_c \partial_d g_{ab} &= 2\partial_c \partial_d g_{ab} + \partial_c \partial_d g_{ab} \\ &= 2g_{eb}\partial_c \Gamma^e_{da} + 2g_{ea}\partial_c \Gamma^e_{db} + g_{eb}\partial_d \Gamma^e_{ca} + g_{ea}\partial_d \Gamma^e_{cb} \\ &= g_{eb}(\partial_c \Gamma^e_{da} - \partial_a \Gamma^e_{cd}) + g_{ea}(\partial_c \Gamma^e_{db} - \partial_b \Gamma^e_{dc}) \\ &= -R_{dbca} - R_{dacb} \end{aligned}$$

using normal coordinates for Riemann tensor (in terms of connection). Just need to tidy up now. We have:

$$\begin{aligned} g_{ab} &= \eta_{ab} - \frac{1}{6}(R_{dbca} + R_{dacb})(x^c - x^c_0)(x^d - x^d_0) \\ &= \eta_{ab} - \frac{1}{6}(R_{dbca} + R_{cadb})\underbrace{(x^c - x^c_0)(x^d - x^d_0)}_{\text{symmetric on } c,d} \quad \text{(can swap } c,\, d\text{)} \\ &= \eta_{ab} - \frac{1}{3}R_{acbd}(x^c - x^c_0)(x^d - x^d_0). \quad \square \end{aligned}$$

3.6 Geodesic deviation

Consider a continuous family of geodesics, labelled by a parameter s, with parameter t along the geodesics. Denote $T^a = \partial x^a/\partial t$ and $S^a = \partial x^a / \partial s$.

Lemma: $T^a \nabla_a S^b = S^a \nabla_a T^b$.

Proof: Writing out in full, we find: $T_a \nabla_a S^b - S^a \nabla_a T^b = T^a \partial_a S^b - S^a \partial_a T^b$. Also:

$$T^a \partial_a S^b = \frac{\partial x^a}{\partial t}\frac{\partial^2 x^b}{\partial x^a \partial s} = \frac{\partial^2 x^b}{\partial t \partial s} = \frac{\partial x^a}{\partial s}\frac{\partial^2 x^b}{\partial x^a \partial t} = S^a \partial_a T^b. \quad \square$$

Theorem: We have the *geodesic deviation equation*:

$$\frac{d^2 S^a}{dt^2} = R^a{}_{bcd} T^b T^c S^d.$$

Proof: We have:

$$\begin{aligned} \frac{d^2 S^a}{dt^2} &= T^c \nabla_c (T^b \nabla_b S^a) = T^c \nabla_c (S^b \nabla_b S^a) \\ &= (T^c \nabla_c S^b)\nabla_b T^a + S^b T^c \nabla_c \nabla_b S^a \\ &= (T^c \nabla_c S^b)\nabla_b T^a + S^b T^c(\nabla_b \nabla_c S^a + R_{cb}{}^a{}_d S^d) \\ &= \cancel{(T^c \nabla_c S^b)\nabla_b T^a} + \underbrace{S^b \nabla_b (T^c \nabla_c T^a)}_{0 \text{ by geodesic eq.}} - \cancel{(S^b \nabla_b T^c)\nabla_c T^a} \\ &\quad + S^b T^c R_{cb}{}^a{}_d S^d. \end{aligned}$$

Using symmetries of Riemann tensor, get result. □

This result shows that *curvature gives rise to 'forces'*: via stretching between geodesics.

3.7 Parallel transport

Definition: Let $x^a(\lambda)$ be a curve, and let $l^a = dx^a/d\lambda$ be its tangent vector. We say that the vector V^a is *parallel-transported* around the curve if $l^b \nabla_b V^a = 0$.

Theorem: Suppose $V^a(\lambda)$ is parallel-transported once around a closed loop, starting and finishing at $\lambda = \lambda_0$. Then the change in $V^a(\lambda)$ is:

$$\Delta V^a = -\frac{1}{2} R^a{}_{cde}(\lambda_0) V^c(\lambda_0) \oint x^d(\lambda') \frac{dx^e}{d\lambda'} d\lambda'.$$

Proof: The parallel transport equation is $l^b \nabla_b V^a$, which can be written out as:

$$\frac{dx^a}{d\lambda}\left(\frac{\partial V^b}{\partial x^a} + \Gamma^b_{ac} V^c\right) = 0.$$

Since $x^a(\lambda)$ is a general curve, need bracket to vanish. Rewrite bracket as the integral equation:

$$V^b(\lambda) = V^b(\lambda_0) - \int_{\lambda_0}^{\lambda} d\lambda' \, \Gamma^b_{ac}(\lambda') \frac{dx^a}{d\lambda'} V^c(\lambda').$$

Taylor-expand the connection and the vector near $\lambda = \lambda_0$, and use the parallel transport equation to substitute for the derivative term $(\partial V^c / \partial x^f)(\lambda_0)$. We have: $V^b(\lambda) =$

$$V^b(\lambda_0) - \int_{\lambda_0}^{\lambda} d\lambda' \, \frac{dx^a}{d\lambda'} \left(\Gamma^b_{ac}(\lambda_0) + \left.\frac{\partial \Gamma^b_{ac}}{\partial x^d}\right|_{\lambda_0} (x^d(\lambda') - x^d(\lambda_0)) \right) \times$$
$$\left(V^c(\lambda_0) - \Gamma^c_{ef}(\lambda_0) V^e(\lambda_0)(x^f(\lambda') - x^f(\lambda_0)) + \ldots\right).$$

Multiply out and evaluate each term as $\lambda \to \lambda_0$ around a closed loop. The lowest order term contains the integral:

$$\oint d\lambda' \frac{dx^a}{d\lambda'} = \oint dx^a = x^a(\lambda_0) - x^a(\lambda_0) = 0.$$

The first order term is:

$$V^e(\lambda_0)\left(\left.\frac{\partial \Gamma^b_{ae}}{\partial x^d}\right|_{\lambda_0} - (\Gamma^c_{ed}\Gamma^b_{ac})(\lambda_0) \right) \oint d\lambda' \frac{dx^a}{d\lambda'} (x^d(\lambda') - x^d(\lambda_0)).$$

Note the integral in this term is antisymmetric under $a \leftrightarrow d$. This follows from integration by parts. Thus we can antisymmetrise over the prefactor in a and d; the prefactor becomes:

$$\frac{1}{2}\left(\left.\frac{\partial \Gamma^b_{ae}}{\partial x^d}\right|_{\lambda_0} - \left.\frac{\partial \Gamma^b_{de}}{\partial x^a}\right|_{\lambda_0} - (\Gamma^c_{ed}\Gamma^b_{ac})(\lambda_0) + (\Gamma^c_{ea}\Gamma^b_{dc})(\lambda_0) \right),$$

which is just $\frac{1}{2} R^b{}_{eda}$. The result follows. □

3.8 The second Bianchi identity

Theorem: $\nabla_e R^a{}_{bcd} + \nabla_c R^a{}_{bde} + \nabla_d R^a{}_{bec} = 0$. This is called the *second Bianchi identity*.

Proof: Using normal coordinates, find that

$$\nabla_c R^a{}_{bcd} = \partial_e \partial_c \Gamma^a_{db} - \partial_e \partial_d \Gamma^a_{cb}.$$

Permute indices and add resulting equations. □

Theorem: $\nabla_e R_{bd} + \nabla_a R^a{}_{bde} - \nabla_d R_{be} = 0$. This is called the *contracted Bianchi identity*.

Proof: Contract on a and c in above. □

Theorem: $\nabla_a \left(R^a{}_b - \frac{1}{2} R \delta^a{}_b\right) = 0$.

Proof: Contract on b and d in above. □

Definition: $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ is called the *Einstein tensor*. By the above Theorem, the derivative of the Einstein tensor is zero: $\nabla^a G_{ab} = 0$.

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4 Action principles for GR

4.1 The Einstein equations

Principle: The equations of general relativity are the *Einstein equations*:

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab},$$

where T_{ab} is a tensor called the *energy-momentum tensor*, determined by the matter present (and constructed below). Λ is a constant called the *cosmological constant*.

Theorem: $\nabla^a T_{ab} = 0$, i.e. the energy-momentum tensor is conserved.

Proof: Since derivatives of Einstein tensor and metric are zero. □

4.2 Integration in curved spacetimes

It would be good to write the Einstein equations as an *action principle*, consistent with other theories in physics. To do so, need to be able to *integrate* in curved space.

Definition: A *tensor density* is an object which transforms as a tensor, up to factors of the Jacobian.

Theorem: The Levi-Civita symbol $\eta_{a_1...a_n}$ is a tensor density.

Proof: The determinant of a matrix obeys:

$$\eta_{b_1 b_2 ... b_n} M^{b_1}{}_{a_1} ... M^{b_n}{}_{a_n} = \eta_{a_1 ... a_n} \det(M).$$

Let $x^a \mapsto \tilde{x}^a$ be a change of coordinates. Set $M^b{}_a = \partial x^b / \partial \tilde{x}^a$, the Jacobian matrix, then

$$\eta_{b_1 b_2 ... b_n} \frac{\partial x^{b_1}}{\partial \tilde{x}^{a_1}} ... \frac{\partial x^{b_n}}{\partial \tilde{x}^{a_n}} = \eta_{a_1 ... a_n} \det\left(\frac{\partial x^a}{\partial \tilde{x}^b}\right)$$

and we're done, by moving the Jacobian to the LHS. □

Theorem: The square root of the determinant of the metric, $\sqrt{|\det(g_{ab})|}$ transforms as a tensor density.

Proof: Use transformation law for metric tensor, then take determinant of both sides. □

Using the above, we can arrange for the Jacobian factors from the Levi-Civita symbol and $\sqrt{\det(g_{ab})}$ to cancel:

Definition: The *alternating tensor* $\epsilon_{a_1...a_n}$ is defined by $\epsilon_{a_1...a_n} = g^{1/2} \eta_{a_1...a_n}$, where $g = \det(g_{ab})$. This transforms as a tensor by the above.

The alternating tensor provides us with a tensorial integration measure:

Theorem: The measure $g^{1/2} d^n x = g^{1/2} dx^1 \wedge ... \wedge dx^n$ is tensorial.

Proof: Notice that $g^{1/2} dx^1 \wedge ... \wedge dx^n =$

$$\frac{1}{n!} g^{1/2} \eta_{a_1...a_n} dx^{a_1} \wedge ... \wedge dx^{a_n} = \frac{1}{n!} \epsilon_{a_1...a_n} dx^{a_1} \wedge ... \wedge dx^{a_n}.$$

Everything on the RHS is tensorial. □

Definition: The *integral* of a scalar Φ in a curved spacetime is defined by:

$$\int \Phi \; g^{1/2} d^n x.$$

This is independent of the coordinates chosen.

Theorem: The alternating tensor has the following properties:

(i) $\epsilon^{a_1 a_2 ... a_p b_1 ... b_{d-p}} \epsilon_{a_1 a_2 ... a_p c_1 ... c_{d-p}} = -p! \delta^{b_1 ... b_{d-p}}_{c_1 ... c_{d-p}}$, where $\delta^{b_1 ... b_{d-p}}_{c_1 ... c_{d-p}}$ is the generalised Kronecker delta, taking the value $+1$ if $c_1 ... c_{d-p}$ are distinct integers, and $b_1 ... b_{d-p}$ is an even permutation of them, -1 if $b_1 ... b_{d-p}$ is an odd permutation of them, and zero otherwise.

(ii) $\nabla_b \epsilon_{a_1 ... a_d} = 0$.

Proof: (i) By definition, $\epsilon_{a_1...a_d} = \sqrt{g} \eta_{a_1...a_d}$. Raising indices:

$$\epsilon^{a_1 ... a_d} = \sqrt{g} g^{a_1 a_1'} ... g^{a_d a_d'} \eta_{a_1' ... a_d'} = -\frac{1}{\sqrt{g}} \eta_{a_1 ... a_d},$$

by a property of the determinant (note $\det(g_{ab}) < 0$). Note the RHS is just treated as a collection of numbers, so the indices do not need to agree.

Thus:

$$\epsilon^{a_1 ... a_p b_1 ... b_{d-p}} \epsilon_{a_1 ... a_p c_1 ... c_{d-p}} = -\eta_{a_1 ... a_p b_1 ... b_{d-p}} \eta_{a_1 ... a_p c_1 ... c_{d-p}}.$$

Considering the possible cases, it is clear this is equal to the given result.

For (ii), simply note that $\nabla_b(\epsilon_{a_1...a_d}) = \nabla_b(\sqrt{g} \eta_{a_1...a_d})$. Note that $\eta_{a_1...a_d}$ are just numbers so can come out of hte covariant derivative, and $\sqrt{g}$ is some function of the metric entries g_{ab}, so its derivative is zero. □

4.3 Stokes' Theorem

Lemma 1: We have the following matrix identity:

$$\frac{d}{dx} \log(\det(M(x))) = \mathrm{tr}\left(M^{-1} M'(x)\right).$$

Proof: Standard result; see Symmetries for proof. □

Lemma 2: $\nabla_a V^a = g^{-1/2} \partial_a (g^{1/2} V^a)$.

Proof: We have: $\nabla_a V^a = \partial_a V^a + \Gamma^a_{ac} V^c$. So need connection Γ^a_{ac}. From the expression for the connection in terms of the metric, we have $\Gamma^a_{ac} = \frac{1}{2} g^{ad} \partial_c g_{ad}$. Write G for the matrix of g_{ab}. Then

$$\Gamma^a_{ac} = \frac{1}{2} \mathrm{tr}\left(G^{-1} \partial_c G\right) = \frac{1}{2} \partial_c \log(\det(G)) = \partial_c (\log(g^{1/2})),$$

where in the last step we just took $\frac{1}{2}$ inside the log. □

Stokes' Theorem: Let Σ be a region in spacetime, and $\partial\Sigma$ its boundary. Let n_a be a unit normal to the boundary, and let γ be the determinant of the metric restricted to $\partial\Sigma$. Then:

$$\int_\Sigma \nabla_a V^a \; g^{1/2} d^n x = \int_{\partial\Sigma} n_a V^a \; \gamma^{1/2} d^{n-1} x.$$

Proof: From the Lemma, the LHS can be written:

$$\int_\Sigma \partial_a(g^{1/2}V^a)\ d^nx.$$

Choose coordinates where x^n is constant on $\partial\Sigma$, and where

$$g_{ab} = \begin{pmatrix}\gamma_{ij} & 0\\ 0 & N^2\end{pmatrix}$$

Define the normal by $n_a = (0, 0, ..., N)$, then raising indices, $n^a = (0, 0, ..., 1/N)$. Integrate over x^n, and replace $g = \gamma N^2$. Then we're left with

$$\int_{\partial\Sigma} V^n N\ \gamma^{1/2} d^{n-1}x = \int_{\partial\Sigma} n_a V^a\ \gamma^{1/2} d^{n-1}x. \quad \square$$

4.4 Another form of Stokes' Theorem

Theorem: Let ω be a p-form. Then

$$\int_\Sigma d\omega = \int_{\partial\Sigma} \omega.$$

Proof: Expand ω in a basis of one-forms:

$$\omega = \frac{1}{p!}\omega_{a_1...a_p} dx^{a_1} \wedge dx^{a_2} \wedge ... \wedge dx^{a_p}.$$

Since $\omega_{a_1...a_p}$ is being contracted with something totally antisymmetric, it itself must be antisymmetric. Hence write it as $\frac{1}{p!}\omega_{a_1...a_p} = \epsilon_{a_1...a_p b}V^b$.

Also, rearrange the wedge product to the form:

$$dx^{a_1} \wedge ... \wedge dx^{a_p} = \eta_{a_1...a_p} dx^1 \wedge ... \wedge dx^p = -\sqrt{\gamma}\epsilon^{a_1...a_p} d^px,$$

where γ is the determinant of the metric restricted to $\partial\Sigma$. Then we have:

$$\int_{\partial\Sigma} \omega = \int_{\partial\Sigma} V^b \epsilon_{a_1...a_p b} dx^{a_1}...dx^{a_p} = (-1)^{p+1}\int_{\partial\Sigma} V^b n_b \gamma^{1/2} d^px,$$

where $n_b = \epsilon_{ba_1...a_p}\epsilon^{a_1...a_p}$. If b is anything other than $p+1$, then $n_b = 0$, so n_b is normal to $\partial\Sigma$ in these coordinates.

Now deal with LHS. Using the form above, we have

$$(d\omega)_{ca_1...a_p} = (p+1)\partial_{[c}\epsilon_{a_1...a_p]b}V^b = (p+1)\nabla_{[c}\epsilon_{a_1...a_p]b}V^b,$$

since this holds in normal coordinates. Recalling $\nabla\epsilon = 0$, we can move the covariant derivative through to get: $(d\omega)_{ca_1...a_p} = (p+1)\epsilon_{[a_1...a_p|b|}\nabla_{c]}V^b$.

There's still work to be done. Let $(d\omega)_{ca_1..a_p} = f\epsilon_{ca_1..a_p}$ for some scalar f. Contracting both sides with $\epsilon^{ca_1...a_p}$ leaves us with $-(p+1)!f$. Doing the same with the expression above:

$$(p+1)\epsilon^{ca_1...a_p}\epsilon_{[a_1...a_p|b|}\nabla_{c]}V^b = (p+1)(-p!)(-1)^p\delta^c{}_b\nabla_c V^b,$$

and so $f = (-1)^p\nabla_b V^b$. Combining this with $dx^c \wedge ... \wedge dx^{a_p} = -\sqrt{g}d^{p+1}x$, we're done by the other form of Stokes' Theorem. $\square$

4.5 The action principle

Definition: The *Einstein-Hilbert action* is defined by:

$$I = \frac{1}{16\pi G}\int_M (R - 2\Lambda)\ g^{1/2}d^4x + \int_M \mathcal{L}_{\text{matter}}\ g^{1/2}d^4x,$$

where $\mathcal{L}_{\text{matter}}$ is some matter Lagrangian.

Lemma: For symmetric M, $\det(M) = \exp(\text{tr}(\log(M)))$.

Proof: M is symmetric, so diagonalisable. Let λ_i be the eigenvalues. Then

$$\det(M) = \lambda_1...\lambda_n = \exp(\log(\lambda_1)+...+\log(\lambda_1)) = \exp(\text{tr}(\log(M))).$$

Hence we're done. $\square$

Theorem: The gravitational part of the action gives rise to the vacuum Einstein equations when extremised.

Proof: Replace g_{ab} by $g_{ab} + h_{ab}$ to vary the action (with h_{ab} infinitesimal). Note $\delta^a{}_b = g^{ac}g_{cb}$ must remain invariant, so $g^{ab} \mapsto g^{ab} - h^{ab}$. Note we raise and lower indices with respect to g_{ab}, the *background metric*.

Now compute how $g^{1/2}$ changes. We have

$$\begin{aligned}\det(g_{ab} + h_{ab}) &= \exp(\text{tr}(\log(g_{ab} + h_{ab})))\\ &= \exp(\text{tr}(\log(g_{ac}) + \log(\delta^c{}_b + h^c{}_b)))\\ &= \exp(\text{tr}(\log(g_{ac})))\exp(\text{tr}(\log(h^c{}_b))) = \det(g_{ac})(1 + h),\end{aligned}$$

where h is the trace of the matrix h_{ab}. Hence variation of $g^{1/2} \mapsto g^{1/2}\left(1 + \frac{1}{2}h\right)$.

Now want variation of R. We have:

$$\delta R = \delta(R_{ab}g^{ab}) = (\delta R_{ab})g^{ba} - R_{ab}h^{ba}.$$

using variation of inverse metric. So need variation in Ricci tensor. Write Ricci tensor using normal coordinates:

$$R_{ce} = \nabla_b\Gamma^b_{ce} - \nabla_e\Gamma^b_{cb}$$

Here, $\nabla_b = \partial_b$ because working in normal coordinates. But this is covariant, so must hold in all coordinates. Take variation:

$$\delta R_{ce} = \nabla_b\delta\Gamma^b_{ce} - \nabla_e\delta\Gamma^b_{cb}.$$

So remains to find variation in Γ. After a short calculation, we find that:

$$\delta\Gamma^a_{bc} = \Gamma^a_{bc}(g+h) - \Gamma^a_{bc}(g) = \frac{1}{2}\left(-\nabla^a h_{bc} + \nabla_b h^a{}_c + \nabla_c h^a{}_b\right).$$

Hence the variation of the Ricci tensor is:

$$\delta R_{ce} = \frac{1}{2}\left(-\nabla_b\nabla^b h_{ce} + \nabla_b\nabla_c h^b{}_e + \nabla_b\nabla_e h^b{}_c - \nabla_e\nabla_c h\right)$$

and so the variation of the Ricci scalar is:

$$\delta R = -\nabla_d \nabla^d h + \nabla_d \nabla_c h^{cd} - R_{ab} h^{ab}.$$

Computing the change in the gravitational part of the action, we find $\delta I_{\text{grav}} =$

$$\frac{1}{16\pi G} \int_M g^{1/2} d^4x \left(\underbrace{-\nabla_d \nabla^d h + \nabla_d \nabla_e h^{de}}_{\text{0 by Stokes' Theorem}} - R_{de} h^{de} + \frac{1}{2} hR - \Lambda h \right).$$

Hence stationary iff $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0$. □

When matter is present, to get the full Einstein equations, we *define*:

Definition: The *energy-momentum tensor* of a matter Lagrangian $\mathcal{L}_{\text{matter}}$ is defined by the variation of the matter part of the Einstein-Hilbert action:

$$\delta I_{\text{matter}} = \int_M \frac{1}{2} T_{ab} h^{ab} \, g^{1/2} d^4x.$$

Note: This definition automatically gives us the Einstein equations. It also forces T_{ab} to be conserved, by the Einstein equations, and also forces T_{ab} to be symmetric (any antisymmetric part would be annihilated by h^{ab}).

Example 1: The energy momentum tensor of a theory with

$$\mathcal{L}_{\text{matter}} = -\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4,$$

is given by:

$$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} \left(g^{cd} \partial_c \phi \partial_d \phi + m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \right).$$

To obtain this, we need to use the fact $\delta \phi = 0$ under variation of the metric (it is called *inert* under variation of the metric).

Example 2: The energy momentum tensor of the electromagnetic Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^{ab} F_{ab},$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is given by:

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}.$$

Here, $\delta A_a = 0$, so A_a is inert.

5 Vacuum solutions of the Einstein equations

5.1 de Sitter spacetime

Consider the vacuum Einstein equations:

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0$$

with $\Lambda > 0$.

Theorem: This equation is equivalent to $R_{ab} = \Lambda g_{ab}$.

Proof: Contract on a, b in Einstein equation. Then $R - \frac{4}{2} R + 4\Lambda = 0$, implying $R = 4\Lambda$. Substituting back into the equations, we're done. □

Let's guess a solution for the metric of the form

$$ds^2 = \Omega^2(t) \left(-dt^2 + dx^2 + dy^2 + dz^2 \right).$$

Computing the Ricci tensor (via very messy calculation, or the conformal transformation theorem - see later), we find that the equation Ω must obey is:

$$\frac{d^2\Omega}{dt^2} = \frac{2}{3} \Lambda \Omega^3.$$

The general solution is $\Omega(t) = A + Bt + \sqrt{\dfrac{3}{\Lambda t^2}}$. Substituting this back into $R_{ab} = \Lambda g_{ab}$, we can determine the constants as $A = B = 0$.

Definition: The spacetime with metric

$$ds^2 = \frac{3}{\Lambda t^2} (-dt^2 + dx^2 + dy^2 + dz^2)$$

is called *de Sitter space*.

As $t \to 0$, we get a singularity. But this is actually an artefact of the coordinates we are using - it is a *coordinate singularity*.

Introduce new coordinates $t = \sqrt{\frac{3}{\Lambda}} e^{T\sqrt{\Lambda/3}}$. The metric becomes:

$$ds^2 = -dT^2 + e^{-2T\sqrt{3/\Lambda}} (dx^2 + dy^2 + dz^2).$$

Over time, the space shrinks, as the spatial part contracts.

We could have chosen $t = \sqrt{\frac{3}{\Lambda}} e^{-T\sqrt{\Lambda/3}}$. Then the metric becomes:

$$ds^2 = -dT^2 + e^{2T\sqrt{3/\Lambda}} (dx^2 + dy^2 + dz^2).$$

So in these coordinates, the space expands! What's going on?

The trouble is our coordinates are *local* - they only describe parts of the spacetime. As an inspired guess, define new coordinates:

$$T = \sqrt{\frac{3}{\Lambda}} \log\left(\frac{V+W}{\sqrt{3/\Lambda}}\right), \quad x = \sqrt{\frac{3}{\Lambda}} \frac{X}{V+W},$$
$$y = \sqrt{\frac{3}{\Lambda}} \frac{Y}{V+W}, \quad z = \sqrt{\frac{3}{\Lambda}} \frac{Z}{V+W},$$

with the condition $-V^2+W^2+X^2+Y^2+Z^2 = 3/\Lambda$. View this as a map $(T,x,y,z) \mapsto (V+W,X,Y,Z)$. The metric becomes:

$$ds^2 = -dV^2 + dW^2 + dX^2 + dY^2 + dZ^2.$$

It is now clear that at constant time ($V =$ const) the spacetime looks like a 3-sphere. In general, a further coordinate transformation given by:

$$V = \sqrt{\frac{3}{\Lambda}} \cosh\left(\tau\sqrt{\frac{\Lambda}{3}}\right), \quad W = \sqrt{\frac{3}{\Lambda}} \sinh\left(\tau\sqrt{\frac{\Lambda}{3}}\right)\cos(\chi),$$
$$X = \sqrt{\frac{3}{\Lambda}} \sinh\left(\tau\sqrt{\frac{\Lambda}{3}}\right)\sin(\chi)\cos(\theta),$$
$$Y = \sqrt{\frac{3}{\Lambda}} \sinh\left(\tau\sqrt{\frac{\Lambda}{3}}\right)\sin(\chi)\sin(\theta)\cos(\phi),$$
$$Z = \sqrt{\frac{3}{\Lambda}} \sinh\left(\tau\sqrt{\frac{\Lambda}{3}}\right)\sin(\chi)\sin(\theta)\sin(\phi),$$

puts the metric in the form:

$$ds^2 = -d\tau^2 + \frac{3}{\Lambda}\cosh^2(3\tau\sqrt{\Lambda})d\sigma_3^2,$$

where $d\sigma_3^2$ is the metric on a 3-sphere. This shows that de Sitter space shrinks to a minimum size of $\sqrt{3/\Lambda}$, and then expands again, indefinitely. The spacetime looks like:

How do we know we've found the whole spacetime? Trial and error.

5.2 Anti-de Sitter spacetime

We can repeat the above calculation for a negative cosmological constant. This time we must guess:

$$ds^2 = \Omega(x)^2(-dt^2 + dx^2 + dy^2 + dz^2).$$

Via very nasty calculation, similar to the above, or via the conformal transformation Theorem from later in the course, we find that Einstein's equations reduce to:

$$\frac{d^2\Omega}{dx^2} = -\frac{2\Lambda}{3}\Omega^3$$

The general solution is: $\Omega(x) = A + Bx + \sqrt{-\frac{3}{\Lambda x^2}}$. Substituting this back into $R_{ab} = \Lambda g_{ab}$, we find $A = B = 0$.

Definition: The spacetime with metric

$$ds^2 = -\frac{3}{\Lambda x^2}(-dt^2 + dx^2 + dy^2 + dz^2)$$

is called *anti-de Sitter space*.

5.3 The Weyl tensor

Definition: The *Weyl tensor* C_{abcd} is the traceless part of the Riemann tensor.

Theorem: In n dimensions, the Weyl tensor is:

$$C_{abcd} = R_{abcd} - \frac{1}{n-2}(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad})$$
$$+\frac{1}{(n-1)(n-2)}R(g_{ac}g_{bd} - g_{ad}g_{bc})$$

Proof: For any symmetric tensor H_{ab} we can define a new tensor H^*_{abcd} satisfying $H^*_{abcd} = -H^*_{abdc}$ and $H^*_{abcd} = H^*_{cdab}$ by:

$$H^*_{abcd} = H_{ac}g_{bd} + H_{bd}g_{ac} - H_{ad}g_{bc} - H_{bc}g_{ad}.$$

Take $H_{ab} = R_{ab}$ and $H_{ab} = g_{ab}$ in turn, and guess:

$$R_{abcd} = C_{abcd} + \alpha R^*_{abcd} + \beta R g^*_{abcd}.$$

Contract on a and c, and impose condition that Weyl tensor is traceless. We get:

$$R_{bd} = \alpha R g_{bd} + \alpha(n-2)R_{bd} + \beta R(2n-2)g_{bd}.$$

Compare coefficients to get result. □

From the above, it is clear that the Weyl tensor has the same symmetries as the Riemann tensor; in addition, it also obeys:

$$C^a{}_{bad} = 0.$$

The interpretation of the Weyl tensor is as follows. Note that R_{ab} is determined by the Einstein equations, from the cosmological constant and the energy-momentum tensor. The Weyl tensor is left undetermined; it tells us the *gravitational degrees of freedom*.

Theorem: The Weyl tensor has $\frac{1}{12}d(d+1)(d+2)(d-3)$ independent components.

Proof: The Weyl tensor has all the symmetries of the Riemann tensor, together with the condition $C^a{}_{bad} = 0$. This constraint is symmetric on the b and d indices, so we get an additional $\frac{1}{2}d(d+1)$ constraints. Hence answer is:

$$\frac{1}{12}d^2(d^2-1) - \frac{1}{2}d(d+1) = \frac{1}{12}d(d+1)(d+2)(d+3). \quad \square$$

Theorem: For de Sitter spacetime, the Riemann tensor is:

$$R_{abcd} = \frac{\Lambda}{3}(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

and the Weyl tensor is zero.

Proof: This follows from a result later in the course. The Weyl tensor is zero in Minkowski space (since $R_{abcd} = 0$ in Minkowski space), and is invariant under conformal transformations, so the Weyl tensor of de Sitter space is zero. Using the definition of the Weyl tensor, and the equations $R_{ab} = \Lambda g_{ab}$, $R = 4\Lambda$, we get the result. □

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6 Conformal transformations

6.1 Causality and geodesics

Definition: A *conformal transformation* of the metric is a transformation of the form $\hat{g}_{ab} = \Omega^2 g_{ab}$, where $\Omega \equiv \Omega(t, x, y, z)$.

Theorem: Conformal transformations preserve the causal structure of the spacetime.

Proof: Clearly if $ds^2 > 0$, $ds^2 = 0$ or $ds^2 < 0$, we have that $\Omega^2 ds^2 > 0$, $\Omega^2 ds^2 = 0$ or $\Omega^2 ds^2 < 0$, respectively. So timelike, null and spacelike separations of points is preserved. □

Theorem: Null geodesics are mapped to null geodesics by conformal transformations.

Proof: Need to work out how connection transforms. Note $g^{ac}g_{cb} = \delta^a{}_b$ must be preserved, so need $g^{ab} \mapsto \Omega^{-2}g^{ab}$. By a short calculation, the connection transforms as:

$$\Gamma^a_{bc}(\hat{g}) = \Gamma^a_{bc}(g) + \frac{1}{\Omega}\left(-g_{bc}\nabla^a\Omega + \delta^a{}_c\nabla_b\Omega + \delta^a{}_b\nabla_c\Omega\right).$$

Note we upgraded to covariant derivatives, because the difference of two connections is a tensor. Hence the geodesic equation transforms to: $0 =$

$$\frac{d^2x^a}{ds^2} + \left(\Gamma^a_{bc}(g) + \frac{1}{\Omega}\left(-g_{bc}\nabla^a\Omega + \delta^a{}_c\nabla_b\Omega + \delta^a{}_b\nabla_c\Omega\right)\right)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

This is horrible for spacelike and timelike geodesics. But for null geodesics, $g_{bc}\dot{x}^b\dot{x}^c = 0$, so this simplifies to:

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}(g)\frac{dx^b}{ds}\frac{dx^c}{ds} = -\frac{2}{\Omega}\frac{dx^a}{ds}\left(\nabla_b\Omega\frac{dx^b}{ds}\right).$$

So maps to a geodesic (although not affine!). Also $g_{bc}\dot{x}^b\dot{x}^c = 0 \mapsto \hat{g}_{bc}\dot{x}^b\dot{x}^c = \Omega^2 g_{bc}\dot{x}^b\dot{x}^c = 0$, so new geodesic is still null. □

6.2 Transformation of tensors

Theorem: Under a conformal transformation in d dimensions, the Riemann tensor, Ricci tensor, Ricci scalar and Weyl tensor transform as $R^a{}_{bcd}(\hat{g}) = R^a{}_{bcd}$

$$\begin{aligned}&+\Omega^{-1}\left(\delta^a{}_d\nabla_c\nabla_b\Omega - g_{bd}\nabla_c\nabla^a\Omega - \delta^a{}_c\nabla_d\nabla_b\Omega + g_{bc}\nabla_d\nabla^a\Omega\right)\\ &+\Omega^{-2}\Big(2\delta^a{}_c\nabla_b\Omega\nabla_d\Omega - 2g_{bc}\nabla^a\Omega\nabla_d\Omega - g_{bd}\delta^a{}_c\nabla_e\Omega\nabla^e\Omega\\ &-2\delta^a{}_d\nabla_b\Omega\nabla_c\Omega + 2g_{bd}\nabla^a\Omega\nabla_c\Omega + g_{bc}\delta^a{}_d\nabla^e\Omega\nabla_e\Omega\Big),\end{aligned}$$

$$\begin{aligned}R_{bd}(\hat{g}) = R_{bd}(g) &+ \Omega^{-1}((2-d)\nabla_b\nabla_d\Omega - g_{bd}\Box_g\Omega)\\ &+\Omega^{-2}(2(d-2)\nabla_b\Omega\nabla_d\Omega + g_{bd}(3-d)\nabla_a\Omega\nabla^a\Omega),\end{aligned}$$

$$\begin{aligned}R(\hat{g}) = \Omega^{-2}R(g) &- 2(d-1)\Omega^{-3}\Box_g\Omega\\ &-(d-1)(d-4)\Omega^{-4}\nabla_a\Omega\nabla^a\Omega,\end{aligned}$$

$$C_{abcd}(\hat{g}) = C_{abcd}(g).$$

Proof: This is a long and messy calculation. From above the connection transforms as:

$$\Gamma^a_{bc}(\hat{g}) = \Gamma^a_{bc}(g) + \left(-g_{bc}\nabla^a\ln(\Omega) + \delta^a{}_c\nabla_b\ln(\Omega) + \delta^a{}_b\nabla_c\ln(\Omega)\right).$$

Hence the Riemann tensor transforms as:

$$R^a{}_{bcd}(g) = \partial_c\Gamma^a_{bd}(g) + \Gamma^a_{ce}(g)\Gamma^e_{bd}(g) - (c \leftrightarrow d) \mapsto$$

$$\begin{aligned}R^a{}_{bcd}(\hat{g}) &= \partial_c\left(\Gamma^a_{bd} + (-g_{bd}\nabla^a\ln(\Omega) + \delta^a{}_d\nabla_b\ln(\Omega) + \delta^a{}_b\nabla_d\ln(\Omega))\right)\\ &+\left(\Gamma^a_{ce} + (-g_{ce}\nabla^a\ln(\Omega) + \delta^a{}_e\nabla_c\ln(\Omega) + \delta^a{}_c\nabla_e\ln(\Omega))\right)\\ &\cdot\left(\Gamma^e_{bd} + (-g_{bd}\nabla^e\ln(\Omega) + \delta^e{}_d\nabla_b\ln(\Omega) + \delta^e{}_b\nabla_d\ln(\Omega))\right) - (c \leftrightarrow d).\end{aligned}$$

Now work in normal coordinates for the original metric; the result is quite easy to derive from there. An intermediate step is:

$$\begin{aligned}R^a{}_{bcd}(\hat{g}) = R^a{}_{bcd}(g) &+ \big(-g_{bd}\nabla_c\nabla^a\ln(\Omega) + \delta^a{}_d\nabla_c\nabla_b\ln(\Omega)\\ &+\delta^a{}_c(\nabla_b\ln(\Omega)\nabla_d\ln(\Omega) - g_{bd}\nabla^e\ln(\Omega)\nabla_e\ln(\Omega))\\ &-g_{bc}\nabla^a\ln(\Omega)\nabla_d\ln(\Omega) - (c \leftrightarrow d)\big).\end{aligned}$$

Contracting indices, we get the Ricci tensor (no inverse metric as a is up and c is down).

To get transformation of R, note $R = R_{ab}g^{ab} \mapsto R_{ab}(\hat{g})\hat{g}^{ab} = R_{ab}(\hat{g})\Omega^{-2}g^{ab}$, and we're home dry.

The Weyl transformation law follows from the above work, by a very bold calculation. □

Theorem: Maxwell's equations are conformally invariant only in four dimensions.

Proof: In GR, Maxwell's equations are: $\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$, $\nabla_a F^{ab} = 0$. Writing both equations out in full:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0,$$
$$\partial_a F^{ab} + \Gamma^a_{ac} F^{cb} = 0.$$

First equation conformally invariant as independent of connection. Recall $\Gamma^a_{ac} = g^{-1/2}\partial_a(g^{1/2})$, then second equation becomes: $\partial_a(g^{1/2}F^{ab}) = 0$.

Under a conformal transformation, $F_{ab} \mapsto F_{ab}$, so $F^{ab} = g^{ac}g^{bd}F_{cd} \mapsto \Omega^{-4}F^{ab}$, and

$$\hat{g}^{1/2} = \det(\hat{g}_{ab})^{1/2} = \Omega^d g^{1/2},$$

where d is the dimension. So equation transforms to $\partial_a(\Omega^{d-4}g^{1/2}F^{ab}) = 0$, and result follows. □

7 Symmetries and Killing vectors

7.1 Definitions and properties

Theorem: The infinitesimal transformation $x^a \mapsto x^a + \xi^a$ is a symmetry if and only if ξ^a obeys *Killing's equation*:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

Proof: A symmetry leaves the line element invariant. So consider:

$$ds^2 = g_{ab}(x+\xi)d(x^a+\xi^a)d(x^b+\xi^b)$$
$$= g_{ab}(x)dx^a dx^b + (\xi^e\partial_e g_{cd} + g_{ad}\partial_c\xi^a + g_{bc}\partial_d\xi^b)dx^c dx^d,$$

by Taylor expanding $g_{ab}(x+\xi)$ and using $d(x^a+\xi^a) = dx^c(\delta^a{}_c + \partial_c\xi^a)$. So we need:

$$\xi^e\partial_e g_{cd} + g_{ad}\partial_c\xi^a + g_{bc}\partial_d\xi^b = 0.$$

Substitute $\partial_c\xi^a = \nabla_c\xi^a - \Gamma^a_{cd}\xi^d$ in the ξ derivatives. Write out connection in terms of metric to see all terms cancel, leaving Killing's equation. □

Definition: Solutions of Killing's equation are called *Killing vectors*.

Theorem: Killing vectors form a Lie algebra.

Proof: The Lie bracket of two Killing vectors k^a, l^a is given by:

$$m^b = [k,l]^b = k^a\nabla_a l^b - l^a\nabla_a k^b.$$

Antisymmetry of bracket is clear. Need closure (i.e. m is a Killing vector), and Jacobi identity (this is hard and we'll assume it can be proved). For closure, we compute:

$$\nabla_a m_b + \nabla_b m_a = \nabla_a k^c\nabla_c l_b - \nabla_a l^c\nabla_c k_b + \nabla_b k^c\nabla_c l_a$$
$$-\nabla_b l^c\nabla_c k_a + k^c\nabla_a\nabla_c l_b - l^c\nabla_a\nabla_c k_b + k^c\nabla_b\nabla_c l_a - l^c\nabla_b\nabla_c k_a.$$

Some terms cancel. Note: $\nabla_a k^c\nabla_c l_b - \nabla_b l^c\nabla_c k_a = \nabla_a k^c\nabla_c l^b - \nabla_c l_b\nabla_a k^c = 0$, using Killing's equation. Similarly $\nabla_a l^c\nabla_c k_b$ cancels with $\nabla_b k^c\nabla_c l_a$. To finish, commute all remaining covariant derivatives past one another:

$$\nabla_a m_b + \nabla_b m_a = \cancel{k^c\nabla_c\nabla_a l_b} + k^c R_{acbd}l^d + \cancel{k^c\nabla_c\nabla_b l_a} +$$
$$k^c R_{bcad}l^d - \cancel{l^c\nabla_c\nabla_a k_b} - l^c R_{acbd}k^d - \cancel{l^c\nabla_c\nabla_b k_a} - l^c R_{bcad}k^d$$

$$= k^c l^d R_{acbd} + k^c l^d R_{bcad} - l^c k^d R_{acbd} - l^c k^d R_{bcad} = 0,$$

using Killing's equation to cancel all second covariant derivatives. Final equality follows from symmetries of Riemann tensor, and the fact we can swap $c \leftrightarrow d$ at will. □

Noether's Theorem: Killing vectors give rise to conserved quantities via the following. Let k^b be a Killing vector, and let u^b be the tangent to a geodesic. Then the quantity $u^b k_b$ is preserved along the geodesic.

Proof: $u^a\nabla_a(u^b k_b) = u^a(\nabla_a u^b)k_b + u^a u^b \nabla_a k_b = 0$, using geodesic equation, and symmetry of $u^a u^b$, antisymmetry of $\nabla_a k_b$. □

Theorem: For any Killing vector, $\nabla_c\nabla_b k_a = R_{abc}{}^d k_d$.

Proof: Note we have $\nabla_a\nabla_b k_c + \nabla_a\nabla_c k_b = 0$ by Killing's equation. Add various forms of this equation, with indices permuted, in a way that we can apply the Bianchi identity: $2R_{abc}{}^d = -R_{bca}{}^d + R_{abc}{}^d - R_{cab}{}^d$. We have:

$$\nabla_b\nabla_c k_a + \nabla_b\nabla_a k_c - \nabla_c\nabla_b k_a - \nabla_c\nabla_a k_b - \nabla_a\nabla_b k_c - \nabla_a\nabla_c k_b$$
$$+\nabla_c\nabla_a k_b + \nabla_c\nabla_b k_a = 0$$
$$R_{bca}{}^d k_d - R_{abc}{}^d k_d + R_{cab}{}^d k_d + \nabla_c\nabla_b k_a - \nabla_c\nabla_a k_b = 0.$$

And hence by Killing's equation and the Bianchi identity, we're left with $\nabla_c\nabla_b k_a = R_{abc}{}^d k_d$. □

7.2 Symmetries of Minkowski spacetime

The Killing vectors of Minkowski spacetime are:

- $k^a = (1,0,0,0)$, corresponding to time translation. The associated conserved quantity is energy.
- $k^a = (0,1,0,0)$, $k^a = (0,0,1,0)$ and $k^a = (0,0,0,1)$ corresponding to spatial translations. The associated conserved triple of quantities is linear momentum.
- $k^a = (0,0,z,-y)$, $k^a = (0,z,0,-x)$ and $k^a = (0,y,-x,0)$ corresponding to spatial rotations. The associated conserved triple of quantities is angular momentum.
- $k^a = (x,t,0,0)$, $k^a = (y,0,t,0)$ and $k^a = (z,0,0,t)$ corresponding to Lorentz boosts. The associated conserved triple of quantities is just the position of the particle at time $t = 0$.

7.3 Killing vectors on a sphere

It's possible to verify that ∂_ϕ and $\sin(\phi)\partial_\theta + \cot(\theta)\cos(\phi)\partial_\phi$ are Killing vectors for the 2-sphere metric $ds^2 = d\theta^2 + \sin^2(\theta)d\phi^2$, via a long, tedious calculation.

To get a third, we use the fact that the Killing vectors form a Lie algebra. The commutator is:

$$m^a = [k,l]^a = k^b\nabla_b l^a - l^b\nabla_b k^a = k^b\partial_b l^a - l^b\partial_b k^a;$$

that is, the commutator is independent of the connection, which is helpful in practice. From here it is easy to find the third Killing vector: $-\cos(\phi)\partial_\theta + \cot(\theta)\sin(\theta)\partial_\phi$.

8 The Newtonian limit

Theorem: The metric $ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2)$, for $\Phi \ll 1$, reproduces Poisson's equation for gravitation via the Einstein equations, and reproduces Newton's law of gravitation via the geodesic equation.

Proof: The non-zero connection components are:

$$\Gamma^t_{ti} = \Gamma^t_{it} = \frac{\partial_i\Phi}{1+2\Phi} \approx \partial_i\Phi, \quad \Gamma^i_{tt} = \frac{\partial_i\Phi}{1-2\Phi} \approx \partial_i\Phi,$$
$$\Gamma^i_{jk} = \frac{1}{1-2\Phi}\left(\delta^i{}_j\partial_k\Phi + \delta^i{}_k\partial_j\Phi - \delta_{jk}\partial^i\Phi\right)$$
$$\approx \delta^i{}_j\partial_k\Phi + \delta^i{}_k\partial_j\Phi - \delta_{jk}\partial^i\Phi.$$

The non-vanishing Ricci tensor components are then:

$$R_{tt} \approx \nabla^2\Phi, \quad R_{ti} \approx 0, \quad R_{ij} \approx -\delta_{ij}\nabla^2\Phi - 2\partial_i\partial_j\Phi.$$

Thus $R = -6\nabla^2\Phi$. Assuming $T_{tt} = \rho$, density, Einstein's equation for the tt component then just gives $\nabla^2\Phi = -4\pi\rho$ as required.

The geodesic equations are:

$$0 \approx \ddot{t} + 2\dot{t}\dot{x}^i\partial_i\Phi,$$
$$0 \approx \ddot{x}^i + \dot{t}^2\partial^i\Phi + 2\dot{x}^i\dot{x}^j\partial_j\Phi - \delta_{jk}\dot{x}^j\dot{x}^k\partial^i\Phi.$$

To simplify, note $\dot{x}^i\partial_i\Phi = \dot{\Phi} = 0$ (for a time-independent gravitational potential), and $\dot{t}^2 - \delta_{ij}\dot{x}^i\dot{x}^j \approx 1 + O(\Phi)$ by the form of the metric.

Then the equations reduce to $\ddot{t} = 0$, i.e. t can be chosen to be proper time, and $0 = \ddot{x}^i + \partial^i\Phi$, i.e. Newton's law of gravitation. □

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9 Tests of general relativity

9.1 The Schwarzschild metric

Definition: The *Schwarzschild metric* is defined by $ds^2 =$

$$-\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).$$

All tests of GR we will see will involve studying motion of particles in this metric.

Theorem: Particles in the Schwarzschild metric have two conserved quantities:

$$E = \left(1-\frac{2M}{r}\right)\dot{t}, \quad L = r^2\sin^2(\theta)\dot{\phi}.$$

Proof: These are clearly first integrals, since the metric is independent of ϕ and t. □

We can get a third conserved quantity as follows:

Theorem: For any geodesic (not necessarily in Schwarzschild), if u^b is a tangent vector, then $u^b u_b$ is conserved along the geodesic.

Proof: $u^a \nabla_a (u^b u_b) = 2u_b u^a \nabla_a u^b = 0$, by geodesic equation. □

In particular, $u^b u_b = g_{ab} u^a u^b$ is conserved. Since this is tensorial, it is independent of coordinates. In normal coordinates, this is $\eta_{ab} u^a u^b$, which is 0 for null geodesics (light) and -1 for timelike geodesics (matter).

So: $-\epsilon = u^b u_b$ is conserved, with $\epsilon = 0$ for light, and $\epsilon = 1$ for matter. In terms of Schwarzschild, this gives:

Theorem: Particles in the Schwarzschild metric have the conserved quantity:

$$-\epsilon = -V\dot{t}^2 + \frac{\dot{r}^2}{V} + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2,$$

where $V = 1 - 2M/r$, and $\epsilon = 0$ for light, 1 for matter.

Proof: Direct from above. □

Theorem: In the Schwarzschild metric, we can without loss of generality set $\theta = \pi/2$.

Proof: Consider the θ equation of motion:

$$\frac{d}{ds}(r^2\dot{\theta}) - r^2\sin(\theta)\cos(\theta)\dot{\phi}^2 = 0.$$

Suppose initially $\theta = \pi/2$, $\dot{\theta} = 0$. Expanding above equation, get $\ddot{\theta} = 0$ initially, so $\theta = \pi/2$ for all time. □

Summary: The following quantities are conserved in the Schwarzschild metric:

$$L = r^2\dot{\phi}, \quad E = \dot{t}V,$$
$$-\epsilon = -V\dot{t}^2 + \frac{\dot{r}^2}{V} + r^2\dot{\phi}^2,$$

where $V = 1 - 2M/r$, and $\epsilon = 0$ for light, 1 for matter. We have set $\theta = \pi/2$ without loss of generality.

9.2 The radial equation

Theorem: We have the radial equation:

$$-\epsilon = \frac{-E^2 + \dot{r}^2}{V} + \frac{L^2}{r^2}.$$

Proof: Eliminate $\dot{\phi}$, $\dot{t}$ from the conserved quantities. □

This has a nice interpretation. Rewrite the radial equation as:

$$\frac{1}{2}\dot{r}^2 + \underbrace{\frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right)}_{V(r)} - \frac{1}{2}E^2 = 0.$$

This should be interpreted as a Newtonian-style energy equation with $V(r)$ the potential energy. Considering $V(r)$ graphically can help us determine things like stability, orbits, and where particles will travel to.

Differentiating the energy equation, we find:

$$\ddot{r} + \underbrace{\frac{\epsilon M}{r^2}}_{\text{inverse square law}} - \underbrace{\frac{L^2}{r^3}}_{\text{centripetal force}} + \frac{3ML^2}{r^4} = 0.$$

This is the same as in Newtonian theory, except with a GR correction $3ML^2/r^4$. We find that for $\epsilon = 0$, the inverse square law vanishes, i.e. light is unaffected by gravity in Newtonian theory.

9.3 The equation of orbital shape

Theorem: The equation of orbital shape is:

$$\frac{d^2u}{d\phi^2} + u - 3Mu^3 - \frac{M\epsilon}{L^2} = 0,$$

where $u(\phi) = 1/r$. Here, $3Mu^2$ is the GR correction to the Newtonian orbital shape equation.

Proof: Use the chain rule:

$$\frac{du}{d\phi} = \frac{du}{dr}\frac{dr}{ds}\frac{ds}{d\phi} = -\frac{1}{r^2}(\dot{r})\frac{r^2}{L} = -\frac{\dot{r}}{L}.$$

Insert into radial energy equation, then differentiate. □

9.4 Satellite delay

Suppose Alice orbits the Earth at height $R + h$, with Bob stationary on its surface, at height R. Alice starts above Bob, orbits once, then looks at her watch to see time t_A has passed. Bob does the same, and looks at his watch to see time t_B has passed.

Claim: The delay is given by $t_A - t_B =$

$$2\pi\sqrt{\frac{(R+h)^3 - 3M(R+h)^2}{M}}\left(1 - \frac{\sqrt{1 - 2M/R}}{\sqrt{1 - 3m/(R+h)}}\right).$$

Proof: Alice is massive and moves in a circular orbit. So using the equation of orbital shape with $u = 1/r =$ constant, and $L = r^2\dot{\phi}$, we find that

$$u - 3Mu^2 - \frac{Mu^4}{\dot{\phi}^2} = 0 \quad \Rightarrow \quad \dot{\phi} = \sqrt{\frac{M}{r^3 - 3Mr^2}}.$$

Hence

$$t_A = 2\pi\sqrt{\frac{(R+h)^3 - 3M(R+h)^2}{M}}.$$

For Bob, $\dot{r} = \dot{\phi} = \dot{\theta} = 0$, and so the Schwarzschild metric gives:

$$t_B = \sqrt{1 - \frac{2M}{R}}\Delta t,$$

where Δt is the change in the coordinate time. Now need coordinate time. Use equation for ϵ to write, using our expression for $\dot{\phi}$:

$$-1 = -V\dot{t}^2 + \left(\frac{M}{r-3M}\right) \quad \Rightarrow \quad \dot{t}^2 = \frac{1}{1-3M/r}.$$

We deduce that $\Delta t = t_A/\sqrt{1-3M/(R+h)}$, and the result follows. □

9.5 Deflection of light rays

Consider a light ray ($\epsilon = 0$) passing the Sun in Newtonian theory:

The orbital equation is $u'' + u = 0$, with boundary condition $u = 0$ when $\phi = 0$. So solution is:

$$u = \frac{\sin(\phi)}{b},$$

which is a straight line with impact parameter b.

In GR, we must work perturbatively. Let u_0 be the Newtonian solution, and let u_1 be a small correction. Substitute into GR orbital shape equation to get:

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3Mu_0^2 = \frac{3M\sin^2(\phi)}{b^2} = \frac{3M}{2b^2}(1-\cos(2\phi)).$$

This is simple and can be solved by a homogeneous solution and a particular integral:

$$\begin{aligned} u_1 &= A\cos(\phi) + B\sin(\phi) + \frac{3M}{2b^2} + \frac{M}{2b^2}\cos(2\phi) \\ &= A\cos(\phi) + B\sin(\phi) + \frac{2M}{b^2} - \frac{M\sin^2(\phi)}{b^2}. \end{aligned}$$

Use BCs such that the photon falls in from the left, $\phi = \pi$, so $u_1(\pi) = 0$. Thus $A = 2M/b^2$. Also WLOG set $B = 0$.

At the right, $\phi = \epsilon$, considered small. Then

$$0 = \frac{\epsilon}{b} + \frac{2M}{b^2} + \frac{2M}{b^2} + O(\epsilon^2).$$

So $\epsilon \approx -4M/b$, so the deflection angle is $4M/b$.

9.6 Precession of planetary orbits

Same problem as light deflection, but now $\epsilon = 1$. Best to try and solve:

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2-1}{L^2} + \frac{2Mu}{L^2} - u^2 + \underbrace{2Mu^3}_{\text{GR bit}}.$$

In the Newtonian case, we get conic sections:

$$u_0 = \frac{M}{L^2}(1 + e\cos(\phi)).$$

For GR, write $u = u_0 + u_1$, where u_1 is a small perturbation. Then to first order we get:

$$\begin{aligned} 2\frac{du_0}{d\phi}\frac{du_1}{d\phi} &= \frac{2Mu_1}{L^2} - 2u_0u_1 + 2Mu_0^3 \\ \Rightarrow -\frac{2Me\sin(\phi)}{L^2}\frac{du_1}{d\phi} + \frac{2M}{L^2}e\cos(\phi)u_1 &= \frac{2M^4(1+e\cos(\phi))^3}{L^6}. \end{aligned}$$

Use an integrating factor to get into form:

$$\frac{d}{d\phi}\left(\frac{u_1}{\sin(\phi)}\right) = \frac{-M^3}{eL^4\sin^2(\phi)}(1+e\cos(\phi))^3.$$

Most terms on the RHS give rise to periodic behaviour in u; however, $3e^2\cos^2(\phi)$ does not! Since:

$$\frac{3e^2\cos^2(\phi)}{\sin^2(\phi)} = \frac{3e^2}{\sin^2(\phi)} - 3e^2,$$

on integrating, we get an interesting term that grows without bound. Considering only this term leaves the simple equation:

$$\frac{d}{d\phi}\left(\frac{u_1}{\sin(\phi)}\right) = \frac{-M^3}{eL^4}(-3e^2) = \frac{-3eM^3}{L^4},$$

Hence:

$$u = \frac{M}{L^2}(1 + e\cos(\phi)) + \frac{3M^3e\phi}{L^4}\sin(\phi) + \text{periodic terms}.$$

This is only the first order term in an approximation. It agrees, to first order, with:

$$u = \frac{M}{L^2}\left(1 + e\cos\left(\phi\left(1 - \frac{3M^2}{L^2}\right)\right)\right).$$

Thus we get a *precessing orbit*. The angle between the distances of closest approach is:

$$\Delta\phi = \frac{2\pi}{1-3M^2/L^2} \approx 2\pi + \frac{6\pi M^2}{L^2}, \quad \text{for } M^2 \ll L^2.$$

This offset from a 2π periodic orbit is called the *precession* of the orbit.

9.7 Gravitational redshift

Consider light ($ds^2 = 0$) emitted radially ($d\theta = d\phi = 0$) from a large spherically symmetric body. The metric reduces to:

$$dt^2 = \frac{dr^2}{V^2} \quad \Rightarrow \quad \Delta t = \int_{r_e}^{r} \frac{dr}{V(r)}.$$

This is the time to propagate in coordinate time from r_e (the radius of emission) and r.

Now let $\Delta\tau_e$ be the proper time interval between successive maxima of the light wave at radius r_e, and $\Delta\tau$ be the proper time interval between successive maxima of the light wave at radius r.

For light, we know $0 = -V\dot{t}^2 + \dot{r}^2/V$. So for an observer at r, $\Delta\tau^2 = V(r)\Delta t^2$, and for an observer at r_e, $\Delta\tau_e^2 = V(r_e)\Delta t^2$. But for our case, we know Δt, and it is equal in both cases. So we find the redshift is:

$$\frac{\Delta\tau_e}{\Delta\tau} = \sqrt{\frac{V(r_e)}{V(r_0)}} \approx 1 - \frac{M}{r_e} + \frac{M}{r}.$$

9.8 Shapiro time delay

Suppose a radio signal is sent from Earth to Venus, which is directly on the opposite side of the Sun, then reflected and sent back to Earth. Let R_V be the Schwarzschild coordinate of Venus and R_E the Schwarzschild coordinate of the Earth. Suppose the radio signal just grazes the surface of the sun, at Schwarzschild coordinate R_0.

Claim: The total travel time of the signal is $T \approx 2(\sqrt{R_E^2 - R_0^2} + \sqrt{R_V^2 - R_0^2})$.

Proof: The radial equation for a null geodesic is

$$0 = \frac{E^2}{V} - \frac{\dot{r}^2}{V} - \frac{L^2}{r^2}.$$

We just graze the Sun, so $r = R_0$ when $\dot{r} = 0$. Hence we find that $E^2/L^2 = (1 - 2M/R_0)/R_0^2$. Using this, and the conserved quantity $E = \dot{t}V$, we find that:

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{R_0^2}{r^2}\frac{1-2M/r}{1-2M/R_0}}.$$

The Sun's Schwarzschild radius, $2M$, is tiny compared to its radius R_0. So assume $1 - 2M/r \approx 1$ throughout. Then:

$$\frac{dr}{dt} \approx \pm\sqrt{1 - \frac{R_0^2}{r^2}}.$$

Since $R_E \gg 2M$, the proper time on Earth is approximately the same as the global coordinate time. So just integrate this between R_E and R_0, then R_0 and R_V (picking the correct minus signs in each case), and double the answer to get result. □

10 Black holes

10.1 Eddington-Finkelstein coordinates

The Schwarzschild metric also describes black holes. This is most easily seen by introducing *Eddington-Finkelstein coordinates*:

Definition: *Ingoing Eddington-Finkelstein coordinates* for the Schwarzschild metric are given by (v, r, θ, ϕ), where

$$v = t + r + 2M\log\left(\frac{r-2M}{2M}\right).$$

Outgoing Eddington-Finkelstein coordinates for the Schwarzschild metric are given by (u, r, θ, ϕ), where

$$u = t - r - 2M\log\left(\frac{r-2M}{2M}\right).$$

Theorem: In ingoing Eddington-Finkelstein coordinates, the Schwarzschild metric takes the form:

$$ds^2 = -V dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).$$

Proof: Short calculation. □

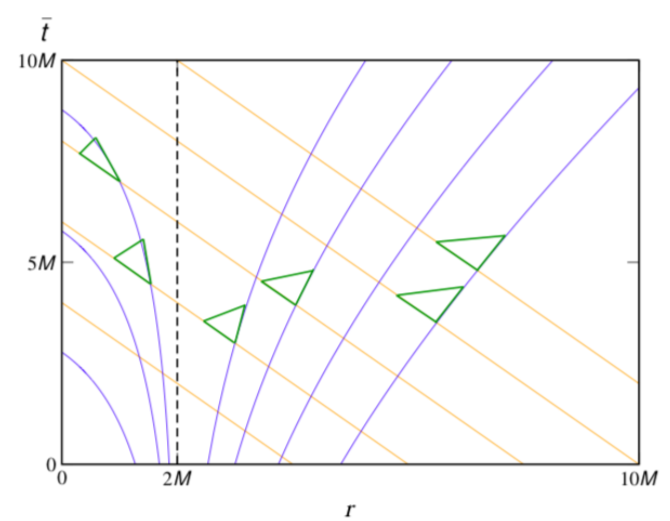
For radial null geodesics, we need either $v =$ constant, or $Vdv = 2dr$, which implies $v - 2r - 4M\log((r-2M)/2M) =$ constant.

For r large, these look like:

constant $= v \approx t + r$, constant $\approx v - 2r \approx t - r$.

So look like Minkowski space lightcones:

As we approach $r \approx 2M$, one of the lines tilts giving us the diagram:



We see that if you are at $r < 2M$, and move along a timelike curve, your r must decrease. So you can never escape if you fall to $r < 2M$.

Definition: We call $r = 2M$ the *event horizon*.

Also note that photons on $r = 2M$ stay there, so you can't see them. This is why it is a *black hole*.

10.2 Properties of black holes

The Hoop Conjecture: If matter of mass M is confined to a proper distance $l \leq 2M$, then an event horizon will form.

Note in Eddington-Finkelstein coordinates, the singularity in Schwarzschild at $r = 2M$ vanishes. But there is still a singularity at $r = 0$. What is its interpretation?

We can see that $r = 0$ is *pathological* by considering the scalar invariant $R_{abcd}R^{abcd}$. It can be shown that:

$$R_{abcd}R^{abcd} \sim \frac{m^2}{r^6},$$

so this scalar blows up in all coordinate systems as $r \to 0$. Classical physics suggests that this singularity is thus the *boundary of spacetime*.

Penrose's Theorem: If there is an event horizon in a spacetime, there is *necessarily* a singularity.

10.3 Hawking radiation

Theorem (Hawking): Particle-anti particle pair creation at the surface of a blackhole means the blackhole acts as a blackbody of temperature $1/8/\pi M$.

Boltzmann's Law: The energy flux of a blackbody per unit area is σT^4, where $\sigma = \pi^2/15$.

Write the energy of the blackhole (equal to its mass in these units) as $M = 1/8\pi T$. Then the specific heat of the black hole is:

$$c = \frac{\partial M}{\partial T} = -\frac{1}{8\pi T^2} < 0.$$

So by the laws of statistical mechanics, black holes are unstable. The black hole breaks down over time. The black hole loses its mass as:

$$\frac{dM}{dt} = -(\text{energy flux}) \propto T^4 M^2 \propto -\frac{1}{M^2}.$$

since $r = 2M$ at the surface of the blackhole, and area $\propto r^2 \propto M^2$. So the black hole's lifetime is $O(M^3)$.

10.4 White holes

The laws of physics are invariant under time reversal. In *outgoing Eddington-Finkelstein coordinates*, as defined above, we find the time-reversed black hole metric:

$$ds^2 = -V du^2 - 2du dr + r^2(d\theta^2 + \sin^2(\theta) d\phi^2).$$

This reverses the diagram above, and shows that all matter inside $r < 2M$ is *ejected* from the region.

Definition: Such a region in spacetime is called a *white hole*.

No one knows why white holes are not observed.

10.5 Photons and black holes

Theorem: There is an unstable circular photon orbit around a black hole at $r = 3M$.

Proof: Use the radial equation of motion, which for a photon is:

$$\ddot{r} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} = 0.$$

For a circular orbit, we need $r =$ constant, which from this equation gives $r = 3M$. To check if this is stable, consider the effective potential energy:

$$V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} - \frac{1}{2}E^2.$$

We find that $V''(3M) < 0$, so the orbit is *unstable*. □

Theorem: The photon absorption cross-section of a black hole is $27\pi M^2$.

Proof: Let a photon be incident on the black hole with impact parameter b. Far away from the black hole, it's as if it isn't there. So set $r = b$, $M = 0$ $\dot{r} = 0$ in the energy equation, to get $b = L/E$. Since these are conserved quantities, this holds as we approach the black hole too.

At $r = 3M$, the maximum of the effective potential energy, the effective potential energy itself is:

$$V(3M) = -\frac{1}{2}E^2\left(1 - \frac{b^2}{27M^2}\right)$$

The photon falls into the black hole if this is less than zero, since if $V(r) < 0$, $\dot{r} < 0$. It follows that the minimum impact parameter a photon can have without being absorbed is $b^2_{\text{min}} = 27M^2$. The absorption cross-section follows. □

10.6 Falling into a black hole

Theorem: For radial free fall into a black hole from a distance $r = 10M$, starting at rest, you would survive a time:

$$T = 5\pi\sqrt{5}M.$$

Proof: The effective potential for a massive particle with no angular momentum is:

$$V(r) = -\frac{1}{2}E^2 + 1 - \frac{2M}{r}.$$

Since starting from rest, $E = 2/\sqrt{5}$. Substituting into the radial equation $\dot{r}^2 + 2V(r) = 0$, we find:

$$\dot{r}\sqrt{\frac{5r}{10M - r}} = -1 \quad \Rightarrow \quad \tau = -\int_{10M}^{0} \sqrt{\frac{5r}{10M - r}}dr,$$

where we take the negative root since we are infalling. Let $r = 10M\sin^2(x)$; integrating then gives result. □

Theorem: With additional angular momentum, it is possible not to fall in. Also, even if you do fall in, the time it takes is longer than the above.

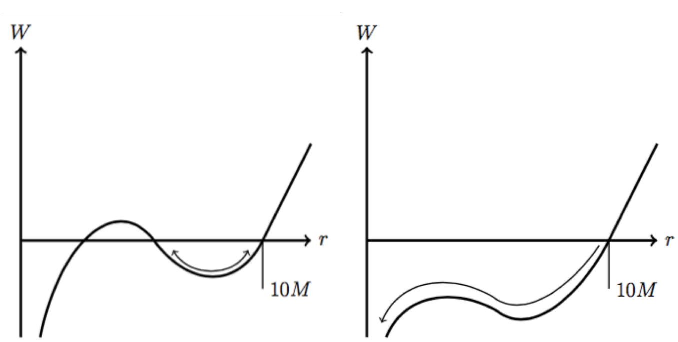
Proof: With angular momentum, the energy becomes:

$$E^2 = \frac{2}{\sqrt{5}}\sqrt{1 + \frac{L^2}{100M^2}}$$

and so the potential may be written:

$$V(r) = \frac{1}{10r^2}\left(1 - \frac{10M}{r}\right)\left(\left(1 - \frac{L^2}{25M^2}\right)r^2 - \frac{2L^2}{5M}r + L^2\right).$$

Considering the roots of this function, we see there are three roots if $2L^2 \geq 25M^2$, and one root if $2L^2 < 25M^2$. These correspond to the graphs:



So if we have sufficient angular momentum, we can oscillate forever in a region outside the black hole. If have more than zero angular momentum, we have a little bump we need to get over in the potential, so it takes longer than if we had no angular momentum. □

11 Cosmology

11.1 FRW metrics

Friedmann, Robertson and Walker developed metrics that describe the whole Universe. It turns out there are three possibilities (derivation not required):

Definition: The *Friedmann, Robertson, Walker metrics* are $ds^2 = -dt^2 + a(t)^2 d\sigma_k^2$. Here, $d\sigma_k^2$ is a spatial metric dependent on a parameter k called the *curvature* taking the values $k = 0, 1$ or -1. The function $a(t)$ is called the *scale factor of the Universe*.

The spatial part of the metric is given by:

$$d\sigma_0^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$
$$d\sigma_1^2 = dr^2 + \sin^2(r)(d\theta^2 + \sin^2(\theta)d\phi^2)$$
$$d\sigma_{-1}^2 = dr^2 + \sinh^2(r)(d\theta^2 + \sin^2(\theta)d\phi^2).$$

Note that for $k = 0, -1$, space is of infinite extent, but for $k = 1$, the metric is just hyperspherical coordinates on S^3, the three-sphere, so space is of finite extent.

11.2 Energy-momentum tensors

The Universe contains stuff so need an energy momentum tensor.

Definition: The energy-momentum tensor of the Universe is

$$T_{ab} = (p + \rho)u_a u_b + p g_{ab},$$

where u^a is a velocity four vector, ρ is energy density and p is pressure. Energy-momentum tensors of this form are said to describe *perfect fluids*.

This energy-momentum tensor captures all possible types of stuff:

- Galaxies, dark matter, are essentially free objects and don't interact very much. Thus they have zero pressure: $p = 0$. Cosmologists call this type of matter *dust*.
- Radiation obeys the equation of state $p = \frac{1}{3}\rho$, from statistical mechanics.
- Dark energy obeys the equation of state $p = -\rho$, as far as anyone can tell experimentally. Note for dark energy, we have $T_{ab} = -\rho g_{ab}$, so the Einstein equations become:

  $$R_{ab} - \frac{1}{2}R g_{ab} + (\Lambda + 8\pi\rho)g_{ab} = 0.$$

  We can thus trade dark energy for a cosmological constant.

11.3 The equations of cosmology

Theorem: We have the mass conservation equation:

$$\dot{\rho} = -\frac{3(p+\rho)\dot{a}}{a}.$$

Proof: Choose $u^a = (1,0,0,0)$ (i.e. comoving coordinates), so that $u_a = (-1,0,0,0)$. Recall $\nabla_a T^{ab} = \nabla_a((p+\rho)u^a u^b + pg^{ab}) = 0$, which in these coordinates gives the result. □

Example: For dust, $p = 0$ so $\rho(t) = \rho_0 a_0^3/a^3(t)$, with $a(t_0) = a_0$, $\rho(t_0) = \rho_0$ for some time t_0. This is conservation of mass:

$$\underbrace{\rho(t)}_{\text{energy density}} \cdot \underbrace{a^3(t)}_{\text{volume}} = \underbrace{\rho_0 a_0^3}_{\text{constant}}.$$

Example: For radiation, $\rho(t) = \rho_0 a_0^4/a^4(t)$, and for dark energy $\rho(t) =$ constant. Thus in an expanding Universe, radiation is diluted faster than matter, which in turn is diluted faster than dark energy. So we expect a radiation-dominated era of the Universe, then a matter-dominated era, then a dark-energy dominated era.

Theorem: We have the *Friedmann* and *Raychaudhuri equations*:

$$4\pi(\rho + 3p) - \Lambda = -3\ddot{a}/a$$
$$3\dot{a}^2 = 8\pi\rho a^2 + \Lambda a^2 - 3k.$$

Proof: Not required. □

We should think of the Raychaudhuri equation as an energy equation:

$$\underbrace{3\dot{a}^2}_{\text{KE}} = \underbrace{8\pi\rho a^2 + \Lambda a^2}_{\text{potential energy}} - \underbrace{3k}_{\text{total energy}}.$$

11.4 Example solutions

Example 1: In the case there is no matter or radiation, i.e. only dark energy. Then we have seen for $\Lambda > 0$ we have de Sitter space, for $\Lambda = 0$ we have Minkowski space, and for $\Lambda < 0$ we have anti-de Sitter space.

Example 2: Suppose there is only dust, and consider a flat Universe. Then $p = k = \Lambda = 0$. This implies

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}.$$

Also $\rho_0 = 1/6\pi t_0^2$, so we can relate density of matter today to the age of the Universe.

At $t = 0$, we find the scale factor $a(t)$ is zero. So there is a singularity (scale can't be zero!).

Penrose's Theorem: In an expanding Universe with no dark energy, there must be a singularity at $t = 0$.

Proof: Not required. □

That is, there is a very early time where this description of the Universe fails.

Example 2: Consider a dust-filled *closed* $(k = 1)$ Universe with no dark energy (i.e. $\Lambda = p = 0$). The Raychaudhuri equation becomes:

$$\dot{a} = \sqrt{\frac{8\pi\rho a_0^3}{3a} - k}.$$

Make the substitution $a(\theta) = 8\pi\rho_0 a_0^3 \sin^2(\theta)/3$. This implies that:

$$\frac{16\pi\rho_0 a_0^3}{3}\int \sin^2(\theta)d\theta = t$$
$$\Rightarrow \quad t(\theta) = \frac{8\pi\rho_0 a_0^3}{3}\left(\theta - \frac{1}{2}\sin(2\theta)\right).$$

So we have a parametric solution. This is a *cycloid*:

We see that the Universe undergoes initial expansion, reaches a maximum size at $\theta = \pi/2$, then contracts. There is a Big Crunch at $t = 8\pi\rho_0 a_0^3/3$.

Example 4: Consider a dust-filled *open* $(k = -1)$ Universe with no dark energy. Via the similar substitution $a(\theta) = 8\pi\rho_0 a_0^3 \sinh^2(\theta)/3$, we have

$$t = \frac{8\pi\rho_a a_0^3}{3}\left(\frac{1}{2}\sinh(2\theta) - \theta\right).$$

There is continuous expansion.

For all cases, we see that at small t, $a(t) \sim t^{2/3}$. However, we've seen that at late times the behaviour is dramatically different:

Since t is small at the moment, it's difficult to know what type of Universe we live in.

Example 5: Consider a radiation-dominated Universe, $p = \frac{1}{3}\rho$, $\Lambda = 0$. Mass conservation gives $\rho(t) = \rho_0 a_0^4 / a^4(t)$ as we saw above. The Raychaudhuri equation becomes:

$$\dot{a} = \sqrt{\frac{8\pi\rho_0 a_0^4}{3a^2} - k}.$$

This can be integrated directly in each of the three k cases, giving:

$$a = \left(\frac{32\pi\rho_0 a_0^4}{3}\right)^{1/4} t^{1/2} \quad (k=0),$$
$$a = \sqrt{2t\left(\frac{8\pi\rho_0 a_0^4}{3}\right)^{1/2} - t^2} \quad (k=1),$$
$$a = \sqrt{t^2 + 2t\left(\frac{8\pi\rho_0 a_0^4}{3}\right)^{1/2}} \quad (k=-1).$$

11.5 Null geodesics in closed universes

Let's find the null geodesics in a $k=1$ Universe. Begin generally: another form of the FRW metric is

$$ds^2 = -dt^2 + a^2\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\right).$$

Defining $dT = \frac{1}{a}dt$, this becomes:

$$ds^2 = a^2\left(-dT^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\right).$$

Under a conformal transformation, null geodesics are preserved. So we may work with metric:

$$ds^2 = -dT^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2.$$

We are interested in $k=1$. Without loss of generality (since the metric is isotropic and homogeneous), let's consider radial null geodesics through $r=0$, with $\theta = \pi/2$. So the metric is:

$$ds^2 = -dT^2 + \frac{dr^2}{1-r^2} + r^2 d\phi^2.$$

From this metric, it's clear we have the conserved quantities:

$$E = -\dot{T} \quad (\text{no } T \text{ dependence})$$
$$L = r^2\dot{\phi} \quad (\text{no } \phi \text{ dependence}).$$

Since goes through $r=0$, $L=0$. Finally, since this is null, we get the equation:

$$0 = E^2 - \frac{\dot{r}^2}{1-r^2} \quad \Rightarrow \quad \tau - \tau_0 = \pm\frac{1}{E^2}\arcsin(r).$$

Hence $r = |\sin(E^2(\tau - \tau_0))|$, so the geodesic returns to its initial point after a time. Since $E = \dot{T}$, we have $r = |\sin(T)|$, so the time light takes to circle the Universe is $\Delta T = \pi$. Inverting, we have

$$\Delta t = \int_0^\pi a(T)dT.$$

This is the time it takes for the light to encircle the Universe. So you can see your younger self if you look off into the distance!

11.6 Einstein's static Universe

Einstein tried to solve the above equations to show the Universe was static. Indeed, for a Universe containing only dust and a cosmological constant, we find that $\dot{a} = \ddot{a} = 0$ if and only if

$$\frac{k}{a_0^2} = 4\pi\rho = \Lambda.$$

Since $\rho > 0$ for dust, $\Lambda > 0$. In a closed Universe, $k=1$, however, this gives an unstable solution when $a(t) = a_0 + \delta a(t)$. We find that $\delta\dot{a}(t) = \Lambda\delta a(t)$, which has exponentially growing solutions.

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12 Gravitational radiation

12.1 Linearised theory

Consider perturbations of spacetime around a background space, so that the metric takes the form $g_{ab} = g^{(0)}_{ab} + h_{ab}$, where h_{ab} is a small perturbation. In general, we will take $g^{(0)}_{ab} = \eta_{ab}$, Minkowski spacetime. Then:

Theorem: The linearised Einstein equations are:

$$-\Box h_{ab} + \partial_d\partial_a h^d{}_b + \partial_d\partial_b h^d{}_a - \partial_a\partial_b h + (\Box h - \partial_d\partial_c h^{cd})\eta_{ab} = 16\pi T_{ab},$$

where $\Box = \nabla_a\nabla^a$ is the d'Alembertian.

Proof: Recall we already calculated the change in the Ricci tensor δR_{ab} and Ricci scalar δR when we considering the Einstein-Hilbert action. We found:

$$\delta R_{ab} = \frac{1}{2}\left(-\Box h_{ab} + \nabla_d\nabla_a h^d{}_b + \nabla_d\nabla_b h^d{}_a - \nabla_a\nabla_b h\right),$$

where $h = h_{ab}g^{(0)^{ab}} = h_{ab}\eta^{ab}$ is the trace of h, and

$$\delta R = -\Box h + \nabla_d\nabla_c h^{cd} - R_{ab}h^{ab}.$$

Assuming there's no matter for the background metric, we have $T_{ab} = O(h)$, so compare the $O(h)$ terms in the Einstein equations, given by:

$$\delta R_{ab} - \frac{1}{2}\delta R g^{(0)}_{ab} - \frac{1}{2}R h_{ab} = 8\pi T_{ab},$$

to get the result. In particular, all covariant derivatives turn into partials because the background is Minkowski. Also note that $R_{ab} = 0$, $R = 0$ in Minkowski spacetime. $\Box$

12.2 Gauge choice

In GR, we have the freedom to make a coordinate transformation $x^a \mapsto x'^a = x^a + \epsilon^a$. If the line element is constant, then we have seen that the metric is changed by $\nabla_a \epsilon_b + \nabla_b \epsilon_a$. However, there are no physical consequences of the coordinate change.

Therefore, in our linearised theory, two metrics h'_{ab}, h_{ab} related by:

$$h'_{ab} = h_{ab} + \partial_a \epsilon_b + \partial_b \epsilon_a$$

are physically equivalent. Transforming between two metrics in this way is called a *gauge transformation*.

Definition: The condition on the metric h'_{ab}

$$\partial_a (h'^{ab} - \frac{1}{2}\eta^{ab} h') = 0,$$

is called *harmonic gauge*.

Theorem: It is always possible to gauge-transform the metric to some h'_{ab} obeying the harmonic gauge condition.

Proof: Let h_{ab} be our initial metric and let h'_{ab} be our transformed metric:

$$h'_{ab} = h_{ab} + \partial_a \epsilon_b + \partial_b \epsilon_a.$$

Then $\partial_a \left(h'^{ab} - \frac{1}{2}\eta^{ab} h' \right) =$

$$\partial_a \left(h^{ab} - \frac{1}{2}\eta^{ab} h \right) + \partial_a \partial^a \epsilon^b + \partial_a \partial^b \epsilon^a - \frac{1}{2}\eta^{ab} \partial_a (2\partial_c \epsilon^c).$$

The last two terms cancel, and we're left with the equation:

$$\Box \epsilon^b = -\partial_a \left(h^{ab} - \frac{1}{2}\eta^{ab} h \right) =: -c^b.$$

So as long as we can solve the equation $\Box \epsilon^b = -c^b$, then we can transform to harmonic gauge.

Let's prove that this equation always has solutions. We do so by constructing the Green's function for $\Box$. Written out in full, the equation $\Box \epsilon^b = -c^b$ is

$$-\frac{\partial^2 \epsilon^b}{\partial t^2} + \nabla^2 \epsilon^b = -c^b.$$

Let the Fourier transform of ϵ^b with respect to both space and time be:

$$\hat{\epsilon}(\mathbf{p}, \omega) = \int d^3\mathbf{x}\, dt\, \epsilon(\mathbf{x}, t) e^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}} \left(= \int d^4x\, \epsilon(x) e^{ip\cdot x} \right).$$

Inverting, we have

$$\epsilon(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int d^3\mathbf{p}\, d\omega\, \hat{\epsilon}(\mathbf{p}, \omega) e^{i\omega t - i\mathbf{p}\cdot\mathbf{x}}.$$

Hence the equation in Fourier space is:

$$\hat{\epsilon}(\mathbf{p}, \omega) = -\frac{\hat{c}(\mathbf{p}, \omega)}{\omega^2 - |\mathbf{p}|^2}.$$

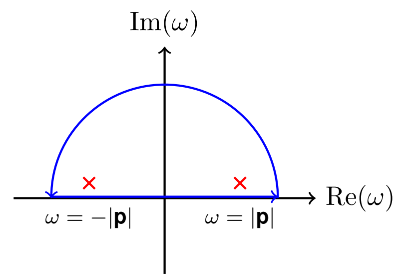
The Green's function, $G(\mathbf{x}, t; \mathbf{x}', t)$ obeys the equation when $-c(\mathbf{x}, t) = \delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t')$, and so

$$\hat{G}(\mathbf{p}, \omega; \mathbf{x}', t') = \frac{e^{-i\omega t' + i\mathbf{p}\cdot\mathbf{x}'}}{\omega^2 - |\mathbf{p}|^2}.$$

Inverting,

$$G(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{(2\pi)^4} \int d^3\mathbf{p}\, d\omega\, \frac{e^{-i\omega t' + i\mathbf{p}\cdot\mathbf{x}'}}{\omega^2 - |\mathbf{p}|^2} e^{i\omega t - i\mathbf{p}\cdot\mathbf{x}}.$$

This integral has poles on the real line at $\omega = \pm|\mathbf{p}|$. We must shift them off in order to carry out the integral. Here, we choose to move *both* of them a tiny bit into the *upper half-plane* (cf. QFT where we move one to the UHP, the other to the LHP). This results in us getting the *retarded Green's function*.



If $t - t' > 0$, close in the UHP, and if $t - t' < 0$, close in the LHP. Then we can apply Jordan's Lemma to say that the contribution from the arc tends to zero as it becomes infinitely large. The residues at the poles are:

$$\mathrm{Res}(\pm|\mathbf{p}|) = \lim_{\omega \to \pm|\mathbf{p}|} \left(\frac{e^{i\omega(t-t') - i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{\omega \pm |\mathbf{p}|} \right) = \pm \frac{e^{\pm i|\mathbf{p}|(t-t') - i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{2|\mathbf{p}|}.$$

So by the residue Theorem, the integral reduces to 0 if $t - t' < 0$, and to

$$\frac{i}{(2\pi)^3} \int d^3\mathbf{p} \left(\frac{e^{i|\mathbf{p}|(t-t') - i\mathbf{p}\cdot\mathbf{r}}}{2|\mathbf{p}|} - \frac{e^{-i|\mathbf{p}|(t-t') - i\mathbf{p}\cdot\mathbf{r}}}{2|\mathbf{p}|} \right).$$

when $t - t' > 0$, where $\mathbf{r} := \mathbf{x} - \mathbf{x}'$. It remains to do this integral, by switching to spherical polars, where $\mathbf{p}\cdot\mathbf{r} = |\mathbf{p}||\mathbf{r}|\cos(\theta) = pr\cos(\theta)$. We find (doing the ϕ integral):

$$\frac{i}{8\pi^2} \int dp\, d\theta\, p \sin(\theta) \left(e^{ip(t-t') - ipr\cos(\theta)} - e^{-ip(t-t') - ipr\cos(\theta)} \right)$$

$$= \frac{1}{8\pi^2 r} \int dp (e^{ip(t-t')} - e^{-ip(t-t')})(e^{ipr} - e^{-ipr})$$

$$= -\frac{1}{2\pi^2 r} \int_0^\infty dp\, \sin(p(t-t')) \sin(pr)$$

$$= \frac{1}{4\pi^2 r} \int_{-\infty}^\infty \frac{1}{2} \left(\cos(p(t - t' + r)) - \cos(p(t - t' - r)) \right)$$

$$= \frac{1}{8\pi^2 r} \mathrm{Re} \left(\int_{-\infty}^\infty e^{ip(t-t'+r)} - e^{-ip(t-t'-)} \right).$$

Performing this final integral, we get

$$\frac{1}{4\pi r}\left(\delta(t-t'+r)-\delta(t-t'-r)\right).$$

The first δ function disappears since $t-t'>0$. So we're left with the final expression for the Green's function:

$$G(\mathbf{x},t;\mathbf{x}',t') = \begin{cases} -\dfrac{1}{4\pi r}\delta(t-t'-r) & \text{for } t-t'>0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the solution to $\Box\epsilon = -c$ is

$$\epsilon(\mathbf{x},t) = \int d^3\mathbf{x}'\ dt'\ G(\mathbf{x},t;\mathbf{x}',t')c(\mathbf{x}',t'). \quad \Box$$

This Green's function tells us something about causality. The δ function here is $\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)$, so $c(\mathbf{x}',t')$ can only affect $\epsilon(\mathbf{x},t)$ in the past lightcone of $(\mathbf{x},t)$. This encodes our desire for *causality*.

Unfortunately, the Green's function as it stands does not look covariant. It can be made covariant using the formula for $\delta(f(x))$; we have:

$$\delta(|x-x'|^2) = \delta((t-t')^2-r^2) = \frac{\delta(t-t'-r)}{2r} - \frac{\delta(t-t'+r)}{2r}.$$

This gives:

$$\epsilon(x) = -\frac{1}{2\pi}\int d^4x\ \delta((x-x')^2)c(\mathbf{x}',t')H(t-t'),$$

where H is the Heaviside function. This is Lorentz invariant, but is not all that useful in practice.

12.3 Wave equation for radiation

Theorem: In harmonic gauge, the linearised Einstein equations reduce to the simple wave equation:

$$\Box h_{ab} = -16\pi\left(T_{ab} - \frac{1}{2}\eta_{ab}T\right),$$

where $T = T^{ab}\eta_{ab}$ is the trace of the energy-momentum tensor.

Proof: In harmonic gauge, we can replace $\partial_d\partial_a h^d{}_b$ in the linearised Einstein equations by $\frac{1}{2}\partial_a\partial_b h$. Thus the equations become:

$$-\Box h_{ab} + \eta_{ab}\Box h - \eta_{ab}\partial_c\partial_d h^{cd} = 16\pi T_{ab}.$$

Again, we can replace $\eta_{ab}\partial_c\partial_d h^{cd}$ by $\frac{1}{2}\eta_{ab}\Box h$ in harmonic gauge. So the equation becomes:

$$-\Box h_{ab} + \frac{1}{2}\eta_{ab}\Box h = 16\pi T_{ab}.$$

Take the trace of this expression by contracting with η_{ab}, the back-substitute. $\Box$

Notice that this is an equation of the form $\Box(\cdot) = ...$, which we spent ages trying to solve when we proved harmonic gauge worked! We can use the same Green's function formula to solve this wave equation for the metric perturbation.

12.4 Polarisation of gravitational waves

Consider first the unsourced equations: $T_{ab}=0$. Then the equations reduce to:

$$\Box h_{ab} = 0, \qquad \partial_a\left(h^{ab} - \frac{1}{2}\eta^{ab}h\right) = 0.$$

Using these equations, we have:

Theorem: Consider a wavelike solution to these equations, $h_{ab} = Ae_{ab}\,\mathrm{Re}(e^{ik_cx^c})$, where A is the amplitude of the perturbation, e_{ab} is the *polarisation tensor*, which must be symmetric since h_{ab} is symmetric. Then we have:

(i) $k_ak^a = 0$, so gravitational waves travel at the speed of light;

(ii) $ik_ae^{ab} = \frac{1}{2}\eta^{ab}ik_ae^c{}_c$, the harmonic gauge condition.

Proof: Just substitute in the ansatz. $\Box$

At first glance, it looks like e_{ab}, being symmetric, has 10 independent components, so gravitational waves have 10 possible polarisations. However, the gauge condition (ii) above gives 4 equations constraining e_{ab}. In fact, there are another 4 constraints on e_{ab} which we will now derive.

Consider a gauge transformation $h_{ab}\to h_{ab}+\partial_a\epsilon_b+\partial_b\epsilon_a$, where $\epsilon_a = -Ai\Lambda_ae^{ik_cx^c}$, so that

$$h_{ab}\to h'_{ab} = h_{ab} + A(\Lambda_ak_b+\Lambda_bk_a)e^{ik_cx^c}.$$

Hence the new theory has the polarisation tensor $e_{ab} = e_{ab}+\Lambda_ak_b+\Lambda_bk_a$. Remarkably though, the harmonic gauge condition is unchanged:

$$k^ae^{ab} - \frac{1}{2}\eta^{ab}k_ae^c{}_c = 0$$

$$\to$$

$$k_ae^{ab} - \frac{1}{2}\eta^{ab}k_ae^c{}_c + \underbrace{k_a(\Lambda^ak^b+\Lambda^bk^a) - \frac{1}{2}\eta^{ab}k_a(2k_c\Lambda^c)}_{=0} = 0.$$

The big brace equals zero by two terms cancelling, and k being null.

Therefore, it is apparent that we have *not fully fixed the metric* by specifying harmonic gauge. There are still four extra degrees of freedom counted by Λ_a. These are degrees of freedom associated with the metric - not with e_{ab}, so e_{ab} has 2 possible polarisations.

Example: Consider a wave travelling in the positive z-direction, i.e. $k^a = k(1,0,0,1)$, $k_a = k(-1,0,0,1)$. Write $x^a = (t,x,y,z)$. Then the exponential in the wave solution is indeed $e^{ik(-t+z)}$, a wave travelling in the z direction at the speed of light.

By fixing the Λ_a, we can fix the possible polarisations of the wave. Under a gauge transformation, we have

$$e_{01} \mapsto e_{01} + \Lambda_0 k_1 + \Lambda_1 k_0 = e_{01} - \Lambda_1 k.$$

Fix $e_{01} = e_{10} = 0$, by appropriate choice of Λ_1 then. Similarly,

$$e_{02} \mapsto e_{02} + \Lambda_0 k_2 + \Lambda_2 k_0 = e_{02} - \Lambda_2 k.$$

Fix $e_{02} = e_{20} = 0$, by appropriate choice of Λ_2. Continuing to e_{03}, we find that

$$e_{03} \mapsto e_{03} + \Lambda_0 k_3 + \Lambda_3 k_0 = e_{03} + \Lambda_0 k - \Lambda_3 k.$$

So we've got two things we can use to fix $e_{03} = e_{30} = 0$. We can use one of them to also set the *trace* of the polarisation tensor to zero:

$$\begin{aligned} e^a{}_a &= -e_{00} + e_{11} + e_{22} + e_{33} \\ &\mapsto -e_{00} + e_{11} + e_{22} + e_{33} - 2\Lambda_0 k_0 + 2\Lambda_1 k_1 + 2\Lambda_2 k_2 + 2\Lambda_3 k_3 \\ &= -e_{00} + e_{11} + e_{22} + e_{33} - 2\Lambda_0 k + 2\Lambda_3 k. \end{aligned}$$

So choose Λ_0, Λ_3 such that *both* $e^c{}_c = 0$ and $e_{03} = 0$.

Finally, examine how this affects the harmonic gauge condition. Recall this is given by $k_a e^{ab} = \frac{1}{2} k^b e^c{}_c = 0$. Hence we have

$$\begin{aligned} b = 0 \;&\Rightarrow\; k_0 e^{00} + k_3 e^{30} = 0 \;\Rightarrow\; e^{00} = 0, \\ b = 1 \;&\Rightarrow\; k_0 e^{01} + k_3 e^{31} = 0 \;\Rightarrow\; e^{13} = 0, \\ b = 2 \;&\Rightarrow\; k_0 e^{02} + k_3 e^{32} = 0 \;\Rightarrow\; e^{23} = 0, \\ b = 3 \;&\Rightarrow\; k_0 e^{03} + k_3 e^{33} = 0 \;\Rightarrow\; e^{33} = 0. \end{aligned}$$

Here, we've used our earlier results, that $e_{03} = e_{01} = e_{02} = 0$. Therefore we are left with only $e_{12} = e_{21}$, and $e_{11} = -e_{22}$ non-zero.

We can separate these into the two possible polarisations:

Definition: The $\times$ polarisation takes $e_{12} \neq 0$. $e_{11} = -e_{22} = 0$, so that

$$e_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The $+$ polarisation takes $e_{11} = -e_{22} \neq 0$ and $e_{12} = 0$, so that

$$e_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Definition: A choice of Λ_a gauge in which $e^c{}_c = 0$ is called *transverse, trace-free* gauge. It's called transverse because the harmonic gauge condition then implies $k_a e^{ab} = 0$, i.e. the polarisation is orthogonal to the direction of travel.

12.5 Gravitational wave detection

In order to detect gravitational waves, we need to look at the relative motion of particles, i.e. we need to examine their geodesic deviation.

Let a family of geodesics be labelled by s, with coordinate t along the geodesics. Define the tangent vector to the geodesics by $U^a = \partial x^a/\partial t$, and the vector taking us between geodesics $V^a = \partial x^a/\partial s$. Recall the geodesic deviation equation:

$$\frac{d^2V^a}{dt^2} = R^a{}_{bcd} U^b U^c V^d \Rightarrow \frac{d^2V^a}{dt^2} + R^a{}_{bcd} U^b U^d V^c = 0.$$

In a particle's rest frame, $U^a = (1,0,0,0)$. Then the equation reduces to:

$$\frac{d^2V^a}{dt^2} = R^a{}_{0c0} V^c = 0.$$

Let's work out the linearised form of $R^a{}_{0c0}$:

Theorem: In transverse trace-free gauge, $R^a{}_{0c0} = -\frac{1}{2}\eta^{ab}\partial_0^2 h_{bc}$.

Proof: Recall the Christoffel symbols are $O(h)$ in perturbation theory, so $\Gamma^2 = O(h^2)$ can be ignored. Thus

$$\begin{aligned} R^a{}_{0c0} &= \partial_c \Gamma^a_{00} - \partial_0 \Gamma^a_{c0} \\ &= \partial_c \left(\frac{1}{2}\eta^{ab}(-\partial_b h_{00} + \partial_0 h_{b0} + \partial_0 h_{b0}) \right) \\ &\quad - \partial_0 \left(\frac{1}{2}\eta^{ab}(-\partial_b h_{0c} + \partial_0 h_{bc} + \partial_c h_{0b}) \right). \end{aligned}$$

In transverse trace-free gauge, $h_{00} = h_{01} = h_{02} = 0 = h_{03}$. So get result. □

Thus the geodesic deviation equation reduces to

$$\frac{d^2V^a}{dt^2} - \frac{1}{2}\eta^{ab}\ddot{h}_{bc} V^c = 0.$$

We deduce that gravitational waves make it appear as if there is a force between the two particles.

Consider two particles in the xy plane now and consider $V^a = (t,x,y,0)$ to be a genuine displacement (with t the displacement in time). Then we get two equations from the geodesic deviation equation:

$$\frac{d^2x}{dt^2} - \frac{1}{2}\ddot{h}_{xx} x - \frac{1}{2}\ddot{h}_{xy} y = 0, \qquad \frac{d^2y}{dt^2} - \frac{1}{2}\ddot{h}_{yx} x - \frac{1}{2}\ddot{h}_{yy} y = 0.$$

For $+$ polarisation, $e_{xx} = 1$ and $e_{yy} = -1$, so these equations reduce to:

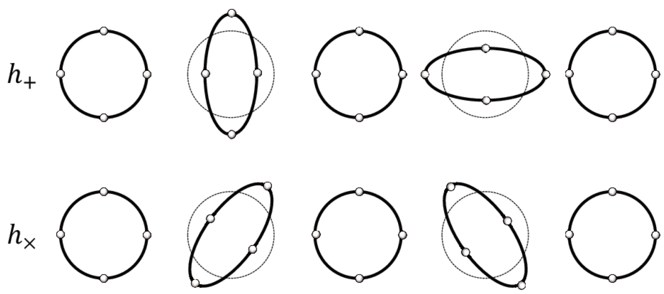
$$\frac{d^2x}{dt^2} - \frac{A}{2}\omega^2\cos(\omega t)x = 0, \qquad \frac{d^2y}{dt^2} + \frac{A}{2}\omega^2\cos(\omega t)y = 0.$$

In $\times$ polarisation, $e_{xy} = e_{yx} = 1$, so these equations reduce to:

$$\frac{d^2x}{dt^2} - \frac{A}{2}\omega^2\cos(\omega t)y = 0, \qquad \frac{d^2y}{dt^2} + \frac{A}{2}\omega^2\cos(\omega t)x = 0.$$

These types of equations are known as *Hill's equations* and are not solvable by elementary methods. However, we can get an idea of the behaviour of the solutions.

Imagine a ring of particles around a fixed particle. Then the two polarisations oscillate the ring as:



12.6 The energy-momentum tensor for gravitational radiation

Recall that we computed the linearisation of the Einstein equations at 1st order. In order to include the fact that the gravitational waves carry energy and momentum, we need to go to second order. The Einstein equations then become:

$$[R_{ab} - \frac{1}{2}Rg_{ab}]^{(1)} = 8\pi T_{ab} - [R_{ab} - \frac{1}{2}Rg_{ab}]^{(2)},$$

where (1), (2) denotes the first order terms, second terms, etc. Therefore the second order terms generate an *effective energy-momentum tensor* in our theory.

Definition: The *effective energy-momentum tensor* is defined by

$$8\pi t_{ab} = -[R_{ab} - \frac{1}{2}Rg_{ab}]^{(2)}.$$

We now just have to calculate this!

Theorem: The second order terms in R_{ab} are:

$$\begin{aligned} R^{(2)}_{ab} = \frac{1}{4}\partial_a h_{cd}\partial_b h^{cd} + \frac{1}{2}h^{cd}\big(\partial_a\partial_b h_{cd} + \partial_c\partial_d h_{ab} - \partial_c\partial_a h_{bd} \\ -\partial_c\partial_b h_{ad}\big) + \frac{1}{2}\partial^d h_b{}^c\left(\partial_d h_{ca} - \partial_c h_{da}\right) \\ -\frac{1}{2}\left(\partial_c h^{cd} - \frac{1}{2}\partial^d h\right)\left(\partial_a h_{db} + \partial_b h_{da} - \partial_d h_{ab}\right). \end{aligned}$$

Proof: Let $g_{ab} + h_{ab}$ be a perturbation to a background spacetime. Let $X[g_{ab}]$ be a quantity dependent on the metric. Then

$$X[g_{ab} + h_{ab}] = X[\eta_{ab}] + \delta X[g_{ab}, h_{ab}] + O(h^2),$$

where δX depends on g_{ab} and h_{ab}. Then taking the variation again, we have:

$$\delta X[g_{ab} + h_{ab}, h_{ab}] = \delta X[g_{ab}] + \delta^2 X[g_{ab}, h_{ab}] + O(h^3).$$

That is, $\delta^2 X[g_{ab}, h_{ab}] = \delta(\delta X))$, the variation of the variation. This can help us calculate the second order terms in h, which are given by $\frac{1}{2}\delta^2 X$ (think of a Taylor series).

We apply this to R_{ab}. We must work with a general metric, then specialise to the Minkowski metric at the end. Recalling that to linear order, $g^{ab} \mapsto g^{ab} - h^{ab}$, we first note

$$\delta\Gamma^a_{bc} = \delta\left(\frac{1}{2}g^{ab}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)\right)$$

$$= -\frac{1}{2}h^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right) + \frac{1}{2}g^{ab}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right)$$

$$= -h_{ed}g^{ae}\Gamma^d_{bc} + \frac{1}{2}g^{ab}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right).$$

Therefore $\delta^2\Gamma^a_{bc} =$

$$\begin{aligned} h_{ed}h^{ae}\Gamma^d_{bc} - h_{ed}g^{ae}\delta\Gamma^d_{bc} - \frac{1}{2}h^{ab}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right) \\ = h_{de}h^{ae}\Gamma^d_{bc} - 2h^a{}_d\delta\Gamma^d_{bc}. \end{aligned}$$

Restricting to Minkowski background, we can set $\Gamma = 0$ in all the above formulas.

Recall that

$$R_{ab} = \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} + \Gamma^d_{ab}\Gamma^c_{cd} - \Gamma^d_{ac}\Gamma^c_{bd}.$$

Thus the variation in R_{ab} is:

$$\delta R_{ab} = \partial_c\delta\Gamma^c_{ab} - \partial_b\delta\Gamma^c_{ac} + \delta\Gamma^d_{ab}\Gamma^c_{cd} + \Gamma^d_{ab}\delta\Gamma^c_{cd} - \delta\Gamma^d_{ac}\Gamma^c_{bd} - \Gamma^d_{ac}\delta\Gamma^c_{bd}.$$

The second variation is (dropping any lone Γ's):

$$\delta^2 R_{ab} = \partial_c\delta^2\Gamma^c_{ab} - \partial_b\delta^2\Gamma^c_{ac} + 2\delta\Gamma^d_{ab}\delta\Gamma^c_{cd} - 2\delta\Gamma^d_{ac}\delta\Gamma^c_{bd}.$$

Now simply substitute in the formulae for the variation in Γ and the second variation of Γ, with $g_{ab} = \eta_{ab}$. Remember to divide by 2 at the end! $\square$

There's still work to be done to get an explicit formula for t_{ab}. For simplicity, we'll assume there's no matter present, i.e. $T_{ab} = 0$. Then we have:

Theorem: In the absence of an energy-momentum tensor, we have

$$t_{ab} = -\frac{1}{8\pi}\left(R^{(2)}_{ab} - \frac{1}{2}\eta^{cd}R^{(2)}_{cd}\eta_{ab}\right).$$

Proof: The first order variation in the Einstein equations is:

$$R^{(1)}_{ab} - \frac{1}{2}\delta(Rg_{ab}) = 0,$$

since there is no matter present. Now $\delta(Rg_{ab}) = h_{ab}R^{(1)}$, since $R = 0$ for Minkowski spacetime. Therefore the first order Einstein equations are $R^{(1)}_{ab} - \frac{1}{2}h_{ab}R^{(1)} = 0$. Contracting on a, b, we get $R^{(1)} - \frac{1}{2}hR^{(1)} = 0$, and since this holds for arbitrary perturbations h, we must have $R^{(1)} = 0$.

Now move to second order variation. The second order variation in the Einstein tensor is:

$$R^{(2)}_{ab} - \frac{1}{4}\delta^2(Rg_{ab}).$$

Evaluating the variation of the second term, we have:

$$\delta^2(Rg_{ab}) = \delta(g_{ab}\delta R + Rh_{ab}) = (\delta^2 R)\eta_{ab} + 2(\delta R)h_{ab} = (\delta^2 R)\eta_{ab},$$

since we found $R^{(1)} = 0$ already. We can also evaluate $\delta^2 R$, as follows: $\delta^2(g^{cd}R_{cd}) =$

$$\delta(-h^{cd}R_{cd} + g^{cd}R^{(1)}_{cd}) = -2h^{cd}R^{(1)}_{cd} + \eta^{cd}R^{(2)}_{cd} = \eta^{cd}R^{(2)}_{cd}.$$

Back-substituting, we get the result. □

This is still rather complicated. But in practice we're not really interested in this quantity itself, but its *average* value, defined as follows:

Definition: Let w be a weight function on a volume V, and let $w\big|_{\partial V} = 0$, $\partial w \ll 1$, and

$$\int_V d^4x\, g^{1/2}\, w = 1.$$

Suppose that the volume has typical size a, and that the variation in X has typical size λ. Then the *average* of X on V is defined by

$$\langle X\rangle = \int_V d^4x\, g^{1/2}\, wX,$$

for $a \gg \lambda$.

Theorem: $\langle\nabla_a Y^a\rangle = 0$, i.e. the average of a total derivative is zero.

Proof: We have: $\langle\nabla_a Y^a\rangle =$

$$\int_V d^4x\, g^{1/2}\, w\nabla_a Y^a = \int_V d^4x\, g^{1/2}\, (\nabla_a(wY^a) - Y^a\partial_a w)$$
$$= \int_{\partial V} d^4x\, g^{1/2}\, wn_a Y^a - \int_V d^4x\, g^{1/2}\, Y^a\partial_a w,$$

using Stokes' Theorem. The first term is zero since w vanishes on the boundary.

The second is negligible, because the components of $\langle\nabla_a Y^a\rangle$ have typical size Y/λ, and the components of the integral on the RHS have typical size $Yw/a\cdot|d^4x| = Y/a$, since the normalisation condition on w implies $|d^4x| \sim 1/w$. Thus the RHS is negligible compared with the LHS. □

This Theorem implies that we can freely integrate by parts when taking an average; that is $\langle A\partial B\rangle \sim -\langle(\partial A)B\rangle$.

With our new averaging notion, we can compute $\langle t_{ab}\rangle$ in a nice form. We have:

Theorem: When there is no energy-momentum tensor present, we have

$$\langle t_{ab}\rangle = \frac{1}{32\pi}\left\langle\partial_a\overline{h}_{cd}\partial_b\overline{h}^{cd} - \frac{1}{2}\partial_a\overline{h}\partial_b\overline{h} - 2\partial_c\overline{h}^c{}_a\partial_d\overline{h}^d{}_b\right\rangle,$$

where $\overline{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h$ is the *trace-reversed* version of the metric h_{ab}.

Proof: We need to find the average of the formula for t_{ab} above. Start with the second term. Begin by noting that

$$g^{1/2}g^{ab}R^{(2)}_{ab} = \frac{1}{2}\delta(g^{1/2}g^{ab}R^{(1)}_{ab}) - \frac{1}{2}R^{(1)}_{ab}\delta(g^{1/2}g^{ab}).$$

Recall $R^{(1)}_{ab} = 0$ when there is no energy-momentum tensor. So the second term drops out.

We found much earlier on that:

$$R^{(1)}_{ab} = \partial_c\delta\Gamma^c_{ab} - \partial_b\delta\Gamma^c_{ac} = \nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac}.$$

The second equality is obtained by going to normal coordinates. Therefore:

$$g^{ab}R^{(1)}_{ab} = \nabla_c\left(g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab}\right).$$

Thus this expression is a total divergence. So $\langle g^{cd}R^{(1)}_{cd}\rangle =$

$$\int_V d^4x\, g^{1/2}g^{cd}R^{(2)}_{cd}w = \frac{1}{2}\delta\left(\int_V d^4x\, g^{1/2}g^{ab}R^{(1)}_{ab}w\right) = 0,$$

since we're integrating a total derivative, so get zero. Set background to $\eta = g$ to finish.

Now we need to deal with the first term, $\langle R^{(2)}_{ab}\rangle$. Integrating by parts, we get:

$$\langle R^{(2)}_{ab}\rangle = \Big\langle -\frac{1}{4}\partial_a h_{cd}\partial_b h^{cd} + \frac{1}{2}\partial^d h^c{}_b(\partial_d h_{ca} - \partial_c h_{da}) + \frac{1}{4}\partial^d h(\partial_a h_{db} + \partial_b h_{da} - \partial_d h_{ab})\Big\rangle$$

Now note that $\overline{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h$ implies that $\overline{h} = -h$, so we can write $h_{ab} = \overline{h}_{ab} - \frac{1}{2}\eta_{ab}\overline{h}$. Substituting this in and doing some algebra we get the result. $\square$

Note that in transverse, trace-free gauge the result simplifies further. In this gauge, we have $\overline{h} = 0$ and $\partial_c \overline{h}^c{}_a = 0$. Therefore we end up with:

Energy-momentum tensor of gravitational wave: In transverse, trace-free gauge, where $\overline{h}$ is the trace-reversed metric, we have

$$\langle t_{ab}\rangle = \frac{1}{32\pi}\langle \partial_a \overline{h}_{cd}\partial_b \overline{h}^{cd}\rangle .$$

The averaging procedure is not just a calculational convenience - it is also essential if we want to get anything physical out of our theory. Indeed, one can check that t_{ab} is *not* gauge invariant (in the sense of changing the metric). Fortunately, we have:

Theorem $\langle t_{ab}\rangle$ is gauge invariant.

Proof: Recall that under a gauge transformation, we have

$$h_{ab} \mapsto h_{ab} + \partial_a\epsilon_b + \partial_b\epsilon_a.$$

Therefore, we have

$$\overline{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}\eta^{cd}h_{cd} \mapsto \overline{h}_{ab} + \partial_a\epsilon_b + \partial_b\epsilon_a - \eta_{ab}\partial_c\epsilon^c.$$

Inserting this into the big formula $\langle t_{ab}\rangle$ (note we can't use the one where $\langle t_{ab}\rangle \sim \langle \partial h \partial h\rangle$, since this fixed a gauge already!), we have that $\langle t_{ab}\rangle \mapsto$

$$\langle t_{ab}\rangle + \frac{1}{32\pi}\big\langle \partial_a \overline{h}^{cd}(\partial_b\partial^c\epsilon^d + \partial_b\partial^d\epsilon^c - \eta^{cd}\partial_d\partial_e\epsilon^e) + \partial_a\overline{h}\partial_b\partial_c\epsilon^c - \partial_c\partial^c\epsilon^d\partial_a\overline{h}_{bd} - \partial_c\overline{h}^{cd}(\partial_a\partial_b\epsilon_d + \partial_a\partial_d\epsilon_b - \eta_{bd}\partial_a\partial_e\epsilon^e) + (a\leftrightarrow b)\big\rangle.$$

Now note that by some calculation, index relabelling and integration by parts, we can reduce this to the form:

$$\langle t_{ab}\rangle + \Big\langle \partial_a\epsilon^c\left(\frac{1}{2}\Box\overline{h}_{bc} - \partial^d\partial_{(b}\overline{h}_{c)d} + \frac{1}{2}\eta_{bc}\partial^d\partial^e\overline{h}_{de}\right) + (a\leftrightarrow b)\Big\rangle.$$

But the big bracket just contains the linearised Einstein equations, which must be satisfied. Thus $\langle t_{ab}\rangle \mapsto \langle t_{ab}\rangle$ as required. $\square$

12.7 Production of gravitational radiation

Let's put some sources, T_{ab}, into the theory now. Recall the equation of motion was:

$$\Box h_{ab} = -16\pi\left(T_{ab} - \frac{1}{2}\eta_{ab}T\right).$$

The equation for the trace-reversed metric (recalling that is even simpler (recalling $\Box h = -16\pi T$, by taking the trace of both sides):

$$\Box\overline{h}_{ab} = -16\pi T_{ab}.$$

This can easily be solved by the standard Green's function to give:

$$\overline{h}_{ab}(\mathbf{x},t) = \int_D d^3\mathbf{x}'\,\frac{4}{|\mathbf{x}-\mathbf{x}'|}T_{ab}(\mathbf{x}',t'),$$

where $t' = t - |\mathbf{x}-\mathbf{x}'|$ and D is some domain which the source is restricted to. We assume in all cases that observers are distant from the source of the radiation, i.e. $|\mathbf{x}| \gg |\mathbf{x}'|$.

We begin by examining *static* cases, $T_{ab}(\mathbf{x}',t') = T_{ab}(\mathbf{x}')$. Since we are distant from the source of the radiation we use, as in electromagnetism, a *multipole expansion*.

Theorem: For $|\mathbf{x}| \gg |\mathbf{x}'|$, we have:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{1}{r^2}r'\cos(\theta) + \frac{1}{2r^3}(r')^2(3\cos^2(\theta)-1) + \cdots,$$

where $r = |\mathbf{x}|$, $r' = |\mathbf{x}'|$ and $\mathbf{x}\cdot\mathbf{x}' = |\mathbf{x}||\mathbf{x}'|\cos(\theta)$.

Proof: We have

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{(r^2+(r')^2-2rr'\cos(\theta))^{1/2}} = \frac{1}{r}\left(1+\frac{(r')^2}{r^2}-\frac{2r'}{r}\cos(\theta)\right)^{-1/2}.$$

Now just use the binomial theorem. $\square$

This gives a nice formula:

$$\overline{h}_{ab} = \frac{4}{r}\int dr'd\theta d\phi\, r'^2\sin(\theta)T_{ab}(\mathbf{x}')\left(1+\frac{r'\cos(\theta)}{r} + \frac{r'^2(3\cos^2(\theta)-1)}{2r^2}+\cdots\right)$$

The first term is the *monopole* term, the second term is the *dipole* term and the third term is the *quadrupole* term.

In practice however, it's usually best to write things in terms of the $\mathbf{x}$ and $\mathbf{x}'$'s, usually with some indices. One of the better forms to work with is:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r}\left(\underbrace{1}_{\text{monopole}} + \underbrace{\frac{\mathbf{x}\cdot\mathbf{x}'}{r^2}}_{\text{dipole}} + \underbrace{\frac{3(\mathbf{x}\cdot\mathbf{x}')^2}{r^4} - \frac{\mathbf{x}'\cdot\mathbf{x}'}{2r^2}}_{\text{quadrupole}}\right).$$

Example: In the static case, we assume that T_{ab} is dominated by its *rest mass energy*. That is, the only non-zero component of T_{ab} is T_{00}. Therefore the only non-zero component of $\bar{h}_{ab}$ is $\bar{h}_{00}$.

Keeping only the monopole, we get:

$$\bar{h}_{00} = \frac{4}{r}\int d^3\mathbf{x}'\; T_{00}(\mathbf{x}') = \frac{4}{r}\int d^3\mathbf{x}'\; \rho(\mathbf{x}') = \frac{4M}{r} = -4\Phi.$$

Here, ρ is the mass density and $\Phi = -M/r$ is the Newtonian gravitational potential. The form of the metric component h_{00} is therefore:

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}(-\bar{h}_{00} + \cancel{\bar{h}_{11}} + \cancel{\bar{h}_{22}} + \cancel{\bar{h}_{33}}) = \frac{1}{2}\bar{h}_{00} = -2\Phi.$$

If we use this as an actual perturbation to Minkowski spacetime, we get something of the form:

$$ds^2 = -(1+2\Phi)\,dt^2 + dx^2 + dy^2 + dz^2,$$

which is the metric reproducing Newtonian gravity.

Including also the dipole term in the expansion, we have:

$$\bar{h}_{00} = \frac{4M}{r} + \frac{4}{r}\int d^3\mathbf{x}'\rho(\mathbf{x}')\frac{\mathbf{x}\cdot\mathbf{x}'}{r^2} = \frac{4M}{r} + \frac{4\mathbf{x}\cdot\mathbf{P}}{r^3},$$

where $\mathbf{P}$ is the *dipole moment*, given by

$$\mathbf{P} = \int d^3\mathbf{x}'\; \rho(\mathbf{x}')\mathbf{x}'.$$

Similarly, we define the *quadrupole moment* to be the matrix:

$$Q_{ij} = \int d^3\mathbf{x}'\; \rho(\mathbf{x}')\left(\frac{3}{2}x'_i x'_j - \frac{1}{2}\delta_{ij}(\mathbf{x}'\cdot\mathbf{x}')\right).$$

12.8 Production of radiation from a time-varying source

We now introduce time-dependence to the source. To deal with time-dependence, we must remember that T_{ab} is conserved: $\partial_a T^{ab} = 0$. Explicitly:

$$\partial_0 T_{00} - \partial_i T_{i0} = 0,$$
$$\partial_0 T_{0j} - \partial_i T_{ij} = 0.$$

Thus if there is time-dependence in T_{00}, we can't neglect T_{i0}, or in turn T_{ij}.

Theorem: $\bar{h}_{00}(\mathbf{x},t) = 4E/r$, where E is the total energy of the source.

Proof: Far away from the source, the radiation looks like gravitational waves with energy-momentum tensor $\langle t_{ab}\rangle \sim \langle(\partial\bar{h})^2\rangle$.

The energy momentum tensor has units of momentum/(area $\times$ time). In terms of energy, this is $\sim E^4 \sim 1/r^4$, since energy is the same as inverse length. Therefore $\partial\bar{h} \sim 1/r^2$. Now $\partial \sim 1/r$, so $\bar{h} \sim 1/r$ in the far field. Therefore, we may restrict to only the monopole term:

$$\bar{h}_{00} = \frac{4}{r}\int d^3\mathbf{x}'\; T_{00}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|) = \frac{4E}{r},$$

where E is the total energy of the source. In principle this is time-varying, but we know from our earlier calculations that changes in energy due to radiation only occur at order $O(h^2)$, so in our linearised theory at the source, this E is constant. □

Theorem: We have

$$\bar{h}_{0i} = -\frac{4}{r}P_i,$$

where P_i is the total momentum of the source.

Proof: This follows immediately from the definition of momentum of a field in QFT, as the integral of T^{0i}. Simply note we had to lower a time index, and thus pick up a minus sign.

Again, we only work to order $1/r$, as in the pervious Theorem. □

Note that since we can always boost to the radiating matter's rest frame, we can always set $P_i = \bar{h}_{0i} = 0$ without loss of generality.

Theorem: We have

$$\bar{h}_{ij} = \frac{2}{r}\ddot{I}_{ij}(t-r),$$

where I_{ij} is the *second moment of the energy density*, given by:

$$I_{ij}(t-r) = \int d^3\mathbf{x}'\, T_{00}x'_i x'_j.$$

Proof: Consider the integral:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} T_{0i}x_j dV = \int_{\mathbb{R}^3}(\partial_k T_{ki})x_j dV$$
$$= \cancel{\int_{\mathbb{R}^2} T_{ki}x_j n_k dS} - \int_{\mathbb{R}^3} T_{ki}\delta_{kj}dV = -\int T_{ij}dV.$$

Now consider the integral:

$$\frac{\partial^2}{\partial t^2}\int T_{00}x_i x_j dV = \frac{\partial}{\partial t}\int \partial_k T_{0k}x_i x_j dV$$
$$= -\frac{\partial}{\partial t}\int (T_{0i}x_j + T_{0j}x_i)dV = 2\int T_{ij}dV,$$

where in the last step we used the previous result. This almost gives the result. We need only recognise that

$$T_{ij}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|) = T_{ij}(\mathbf{x}', t-|\mathbf{x}'|) + O(1/r),$$

when we Taylor expand T_{ij} for $|\mathbf{x}| \gg |\mathbf{x}'|$. So any higher order terms don't feature, as we already have the $1/r$ out front. □

Theorem: We can rewrite $\bar{h}_{ij}$ in terms of the quadrupole tensor as

$$\bar{h}_{ij} = \frac{4}{3r}\frac{\partial^2 Q_{ij}}{\partial t^2}.$$

Proof: Quick calculation. We get the trace of Q_{ij} at some point, but it's easy to see this is zero straight from its definition. □

It's possible to substitute all this in to our formula for $\langle t_{ab}\rangle$ and eventually get a formula for the power per unit area carried away from the source.

Integrating over a sphere, we eventually get that:

Theorem: The total power from a source via gravitational waves is given by the *quadrupole formula*

$$\text{total power} = \frac{4}{45}\dddot{Q}_{ij}\dddot{Q}_{ij}.$$

12.9 Example calculations

Example 1 (Emission of radiation from two orbiting particles): Consider two bodies with equal mass m, moving in a single circular orbit. The bodies have positions:

$$\mathbf{x}_1 = (r\cos(\psi), r\sin(\psi), 0), \quad \mathbf{x}_2 = (-r\cos(\psi), -r\sin(\psi), 0),$$

i.e. are diametrically opposite. The angular velocity of the particles is given by

$$\dot{\psi}(t) = \sqrt{\frac{2m}{r}}.$$

Recall that the energy-momentum tensor for a particle with momentum p^μ at position $\mathbf{x}(t)$ is given by

$$T^{ab}(\mathbf{x}) = \frac{p^a p^b}{p^2}\delta(\mathbf{x}-\mathbf{x}(t)).$$

For two particles, we just sum two energy-momentum tensors of this type. Note also that in the non-relativistic limit, which we'll assume here, that $p^a = (m, mv^i)$, where v^i is the velocity of the particle.

Thus in our case the energy density is:

$$T^{00} = m\delta(\mathbf{x}-\mathbf{x}_1) + m\delta(\mathbf{x}-\mathbf{x}_2).$$

This allows us to compute the second moment of the energy distribution:

$$I^{ij} = \int d^3\mathbf{x}T^{00}(\mathbf{x})x^i x^j = mx_1^i x_1^j + mx_2^i x_2^j.$$

The quadrupole moment is then

$$Q_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I_{kk} = mx_1^i x_1^j + mx_2^i x_2^j - \frac{1}{3}\delta_{ij}mr^2.$$

Writing out the second moment of the energy distribution in gory detail, we have:

$$I_{ij} = 2mr^2\begin{pmatrix}\cos^2(\psi) & \cos(\psi)\sin(\psi)\\ \cos(\psi)\sin(\psi) & \sin^2(\psi)\end{pmatrix}.$$

We now need to take the time derivative a few times. After a calculation, we eventually find that the total power of the gravitational waves emitted by this system is

$$\frac{64}{5}\cdot\frac{m^5}{r^5}.$$

13 Vierbein fields

13.1 Definitions and basic properties

Since the metric g_{ab} is symmetric, and has signature $\{-1,+1,+1,+1\}$, it can be diagonalised.

Definition: Write $g_{ab} = e^{\mu}{}_{a}e^{\nu}{}_{b}\eta_{\mu\nu}$. We call the matrices $e^{\mu}{}_{a}$, $e^{\nu}{}_{b}$ the *vierbein* or *frame fields*.

Note that the Greek index of a vierbein is a *Lorentz index* so it is raised and lowered wrt $\eta_{\mu\nu}$, whilst the Latin index is a *spacetime index*, so it raised and lowered wrt g_{ab}. Therefore, we define things like

$$e_{\mu a} = \eta_{\mu\nu}e^{\nu}{}_{a}, \quad e^{a}{}_{\mu} = g^{ab}\eta_{\mu\nu}e^{\nu}{}_{b}, \quad \text{etc.}$$

A useful consequence of this is:

Theorem: $e^{a}{}_{\mu}e^{\mu}{}_{b} = \delta^{a}{}_{b}$ and $e^{\mu}{}_{a}e^{a}{}_{\nu} = \delta^{\mu}{}_{\nu}$.

Proof: Simply recall the definition of the fields:

$$e^{a}{}_{\mu}e^{\mu}{}_{b} = g^{ac}\eta_{\mu\nu}e^{\nu}{}_{c}e^{\mu}{}_{b} = g^{ac}g_{cb} = \delta^{a}{}_{b}.$$

For the second equation, invert the vierbein matrices in the defining equation: $e^{a}{}_{\mu}g_{ab}e^{b}{}_{\nu} = \eta_{\mu\nu}$. Raising a μ and lowering a b, we get the result. □

The vierbein fields help us to identify the following symmetry:

Theorem: Under a Lorentz transformation of the vierbeins, the metric is invariant.

Proof: We have $e^{\mu}{}_{a} \mapsto \tilde{e}^{\mu}{}_{a} = \Lambda^{\mu}{}_{\nu}e^{\nu}{}_{a}$ under a Lorentz transformation. hence

$$g_{ab} \mapsto \tilde{e}^{\mu}{}_{a}\tilde{e}^{\nu}{}_{b}\eta_{\mu\nu} = \Lambda^{\mu}{}_{\sigma}e^{\sigma}{}_{a}\Lambda^{\nu}{}_{\rho}e^{\rho}{}_{b}\eta_{\mu\nu} = e^{\sigma}{}_{a}e^{\rho}{}_{b}\eta_{\sigma\rho} = g_{ab},$$

using $\Lambda^{T}\eta\Lambda = \eta$. □

This is a *hidden symmetry* of the metric that we have identified using the vierbeins. In fact, the Lorentz transformation can even vary from spacetime point to spacetime point.

Slogan: We say that the *local* Lorentz transformation form a hidden symmetry of the metric.

The line element in terms of the vierbein fields is

$$ds^2 = g_{ab}dx^a dx^b = \eta_{\mu\nu}e^{\mu}{}_{a}dx^a e^{\nu}{}_{b}dx^b = \eta_{\mu\nu}E^{\mu}E^{\nu},$$

where $E^{\mu} := e^{\mu}{}_{a}dx^a$. The $\{E^{\mu}\}$ form a basis of one-forms, thus the vierbeins allow us to write the line element in a Lorentzian way. This also shows the vierbeins are components of a one-form.

Note that since $ds^2 = \eta_{\mu\nu}E^{\mu}E^{\nu}$, the $\{E^{\mu}\}$ form a *pseudo-orthonormal basis* (to be orthonormal, we'd need to replace $\eta_{\mu\nu} \mapsto \delta_{\mu\nu}$ in the line element).

Objects like $e^{\mu}{}_{a}V^a$, where V^a is a vector, are scalars wrt their spacetime indices (since all spacetime indices are contracted), but vectors wrt their Lorentz indices. Thus $V^{\mu} := e^{\mu}{}_{a}V^a$ is a *Lorentz vector*.

Inverting, we have $e^{b}{}_{\mu}V^{\mu} = e^{b}{}_{\mu}e^{\mu}{}_{a}V^a = \delta^{b}{}_{a}V^a = V^b$. Hence $V^a = e^{a}{}_{\mu}V^{\mu}$ is a *spacetime vector*.

13.2 Derivatives

We now want to define the covariant derivative with a Lorentz index. Since $\partial_{\mu} = e^{a}{}_{\mu}\partial_a$, this is easy for scalars, and we can simply write $\nabla_{\mu}\phi = \partial_{\mu}\phi$.

For vectors, we need to be more careful...

Definition: The *covariant derivative* with a Lorentz index acting on a Lorentz vector is defined by

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \omega_{\mu}{}^{\nu}{}_{\rho}V^{\rho}.$$

The connection ω is called the *spin connection*. In the usual way (i.e. construct a scalar $S = V^{\nu}V_{\nu}$ and act on it with the covariant derivative), we can extend this to one-forms via:

$$\nabla_{\mu}V_{\nu} = \partial_{\mu}V_{\nu} - \omega_{\mu}{}^{\rho}{}_{\nu}V_{\rho}.$$

Definition: In analogy with the metric connection, we impose the condition $\nabla_a e^{b}{}_{\nu} = 0$ on the covariant derivative. This condition is called the *vierbein postulate*.

Theorem: The vierbein postulate is equivalent to

$$\partial_a e^{b}{}_{\nu} + \Gamma^{b}{}_{ac}e^{c}{}_{\nu} - e^{\mu}{}_{a}\omega_{\mu}{}^{\rho}{}_{\nu}e^{b}{}_{\rho} = 0.$$

Proof: Notice we're trying to differentiate an object with one Lorentz index and one spacetime index. Let's work out how to do this.

Notice that

$$\nabla_a(e^{b}{}_{\nu}V^{\nu}) = \partial_a(e^{b}{}_{\nu}V^{\nu}) + \Gamma^{b}{}_{ac}e^{c}{}_{\nu}V^{\nu}$$

holds for any V^{ν}, since we're trying to differentiate a spacetime vector. Using the Leibniz rule on the LHS, we have

$$V^{\nu}\nabla_a(e^{b}{}_{\nu}) + e^{b}{}_{\nu}\nabla_a V^{\nu} = V^{\nu}\nabla_a(e^{b}{}_{\nu}) + e^{b}{}_{\nu}e^{\mu}{}_{a}\nabla_{\mu}V^{\nu}.$$

Now use the definition of the covariant derivative of a Lorentz vector in terms of the spin connection. Rearranging everything we've found already, we see that:

$$V^{\nu}\nabla_a(e^{b}{}_{\nu}) = V^{\nu}\partial_a(e^{b}{}_{\nu}) + \Gamma^{b}{}_{ac}e^{c}{}_{\nu}V^{\nu} - e^{b}{}_{\nu}e^{\mu}{}_{a}\omega_{\mu}{}^{\nu}{}_{\rho}V^{\rho}.$$

Swapping some indices around, the result follows. □

Theorem: Assuming the vierbein postulate, the spin connection takes the explicit form:

$$\omega_{\lambda\tau\nu} = e^a{}_\lambda e_{b\tau}\left(\partial_a e^b{}_\nu + \Gamma^b{}_{ac}e^c{}_\nu\right) = e^a{}_\lambda e_{b\tau}\nabla_a e^b{}_{(\nu)},$$

where the brackets mean miss out the Lorentz index when taking the covariant derivative.

Proof: Rearranging the form of the vierbein postulate we found above, we see that

$$\partial_a e^b{}_\nu + \Gamma^b{}_{ac}e^c{}_\nu = e^\mu{}_a \omega_\mu{}^\rho{}_\nu e^b{}_\rho.$$

Now just invert the vierbein fields on the RHS by multiplying through by $e^a{}_\lambda$ and $e_{b\tau}$. $\square$

Again mirroring our earlier work, we may also require the Lorentzian *torsion* to vanish:

Definition: The covariant derivative is called *torsion-free* if

$$\nabla_\mu e^{(a)}{}_\nu - \nabla_\nu e^{(a)}{}_\mu = T^\rho_{\mu\nu} e^a{}_\rho = 0.$$

The brackets mean we do not include the spacetime indices when taking the covariant derivative. We call $T^\rho_{\mu\nu}$ the *torsion tensor*.

Theorem: The torsion tensor vanishes if $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$.

Proof: First note that: $\nabla_\mu e^{(a)}{}_\nu =$

$$e^b{}_\mu \nabla_b e^{(a)}{}_\nu = e^b{}_\mu\left(\partial_b e^a{}_\nu - e^\sigma{}_b \omega_\sigma{}^\rho{}_\nu e^a{}_\rho\right) = e^b{}_\mu \Gamma^a{}_{bc} e^c{}_\nu.$$

by the earlier work. Therefore the difference is $e^b{}_\mu \Gamma^a{}_{bc} e^c{}_\nu - e^b{}_\nu \Gamma^a{}_{bc} e^c{}_\mu = e^b{}_\mu e^c{}_\nu (\Gamma^a{}_{bc} - \Gamma^a{}_{cb})$. $\square$

Finally, another analogy to the metric connection in this context would be $\nabla_\mu \eta_{\nu\rho} = 0$. What does imposing this condition do?

Theorem: $\nabla_\mu \eta_{\nu\rho} = 0$ is equivalent to $\omega_{\mu\rho\sigma} = -\omega_{\mu\sigma\rho}$, i.e. antisymmetry of the spin connection.

Proof: We have: $0 = \nabla_\mu \eta_{\nu\rho} = -\omega_\mu{}^\sigma{}_\nu \eta_{\sigma\rho} - \omega_\mu{}^\sigma{}_\rho \eta_{\nu\sigma}$, so done. $\square$

13.3 Cartan's first equation of structure

To apply all of this to calculations, we need some extra machinery. Recall our basis of one forms was defined by $E^\mu = e^\mu{}_a dx^a$. Taking the exterior derivative, we have

$$dE^\mu = \frac{1}{2} c^\mu{}_{\nu\rho} E^\nu \wedge E^\rho,$$

for some coefficients $c^\mu{}_{\nu\rho}$.

Definition: The coefficients $c^\mu{}_{\nu\rho}$ are called the *Ricci rotation coefficients*.

Theorem: We have $\omega_{\mu\nu\rho} = \frac{1}{2}(-c_{\mu\nu\rho} - c_{\nu\mu\rho} + c_{\rho\mu\nu})$.

Proof: Recall that $dE^\mu = \partial_b e^\mu{}_a dx^b \wedge dx^a$, so we're left with:

$$dE^\mu = \partial_b e^\mu{}_a dx^b \wedge dx^a = e^a{}_\nu \partial_b e^\mu{}_a dx^b \wedge dE^\nu.$$

Now notice that $0 = \partial_b(\delta^\mu{}_\nu) = e^\mu{}_a \partial_b e^a{}_\nu + e^a{}_\nu \partial_b e^\mu{}_a$. Using this result, rewrite dE^μ as:

$$dE^\mu = -e^\mu{}_a \partial_b e^a{}_\nu dx^b \wedge dE^\nu = e^b{}_\rho e^\mu{}_a \partial_b e^a{}_\nu dE^\nu \wedge dE^\rho.$$

Recall that

$$\omega_\rho{}^\mu{}_\nu = e^b{}_\rho e^\mu{}_a\left(\partial_b e^a{}_\nu + \Gamma^a{}_{bc} e^c{}_\nu\right)$$

Notice that under exchange of ν, ρ in the second term of this formula, we have:

$$e^b{}_\rho e^\mu{}_a \Gamma^a{}_{bc} e^c{}_\nu \mapsto e^b{}_\nu e^\mu{}_a \Gamma^a{}_{bc} e^c{}_\rho$$

But relabelling $c \leftrightarrow b$, we get the same thing as we had originally, because Γ is symmetric on its downstairs indices (assuming a torsion free connection). Therefore:

$$dE^\mu = \omega_\rho{}^\mu{}_\nu dE^\nu \wedge dE^\rho.$$

It follows that $\frac{1}{2} c_{\mu[\nu\rho]} = \omega_{[\rho|\mu|\nu]}$. Writing this out explicitly, we have:

$$\frac{1}{2}\left(c_{\mu\nu\rho} - c_{\mu\rho\nu}\right) = c_{\mu\nu\rho} = \omega_{\rho\mu\nu} - \omega_{\nu\mu\rho}.$$

We get the second equality, because by definition, c is anti-symmetric on its second and third indices. Writing this out three times with $\mu\nu\rho \mapsto \nu\mu\rho \mapsto \rho\mu\nu$, we have:

$$\begin{aligned} c_{\mu\nu\rho} - c_{\nu\mu\rho} + c_{\rho\mu\nu} &= \omega_{\rho\mu\nu} - \omega_{\nu\mu\rho} - \omega_{\rho\nu\mu} + \omega_{\mu\nu\rho} + \omega_{\nu\rho\mu} - \omega_{\mu\rho\nu} \\ &= 2\left(\omega_{\mu\nu\rho} + \omega_{\nu\rho\mu} + \omega_{\rho\mu\nu}\right). \end{aligned}$$

Finally, $\omega_{\nu\rho\mu} + \omega_{\rho\mu\nu} = -\omega_{\nu\mu\rho} + \omega_{\rho\mu\nu} = c_{\mu\nu\rho}$ and hence we find

$$\omega_{\mu\nu\rho} = \frac{1}{2}(-c_{\mu\nu\rho} - c_{\nu\mu\rho} + c_{\rho\mu\nu}). \quad \square$$

We can extend this result to a torsionful connection. First we define:

Definition: The *connection* 1*-form* is defined by $\omega^\mu{}_\nu = \omega_\rho{}^\mu{}_\nu E^\rho$. The *torsion* 2*-form* is defined by $\Theta^\mu = \frac{1}{2} T^\mu_{\nu\rho} E^\nu \wedge E^\rho$.

Theorem (Cartan's 1st eqn. of structure): We have:

$$dE^\mu + \omega^\mu{}_\nu \wedge E^\nu = \Theta^\mu.$$

Proof: The result follows from the proof above. This time, we don't assume the connection is torsion-free, so we get up to the stage:

$$dE^\mu = e^b{}_\rho e^\mu{}_a \partial_b e^a{}_\nu dE^\nu \wedge dE^\rho.$$

Replacing the derivative by the spin connection and connection, we have:

$$dE^\mu = \left(\omega_\rho{}^\mu{}_\nu - e^b{}_\rho e^\mu{}_a \Gamma^a{}_{bc} e^c{}_\nu\right) E^\nu \wedge E^\rho.$$

Recall from our torsion calculation that

$$e^b{}_\rho \nabla_b e^{(a)}{}_\nu = e^b{}_\rho \Gamma^a{}_{bc} e^c{}_\nu$$

Inserting this into the above we have:

$$dE^\mu = \left(\omega_\rho{}^\mu{}_\nu - e^\mu{}_a e^b{}_\rho \nabla_b e^{(a)}{}_\nu\right) E^\nu \wedge E^\rho.$$

By antisymmetry of ν, ρ imposed by the wedge product, can replace the second term by:

$$\frac{1}{2}\left(e^\mu{}_a e^b{}_\rho \nabla_b e^{(a)}{}_\nu - e^\mu{}_a e^b{}_\nu \nabla_b e^{(a)}{}_\rho\right) = \frac{1}{2} e^\mu{}_a T^\sigma_{\rho\nu} e^a{}_\sigma = \frac{1}{2} T^\mu_{\rho\nu}.$$

It's now clear how to obtain the result. □

13.4 The curvature tensor

Continuing on our tour of differential geometry, we can define the Riemann curvature tensor with Lorentzian indices as follows:

Definition: $R_{\mu\nu\sigma}{}^\rho(\omega) V_\rho := (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\sigma$.

Definition: The *curvature 2-form* is defined by

$$\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma}(\omega) E^\rho \wedge E^\sigma.$$

Theorem (Cartan's 2nd eqn.): We have:

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu.$$

Proof: Begin by noting:

$$\omega_a{}^\mu{}_\nu = e^\mu{}_b e^c{}_\nu \Gamma^d{}_{ac} - e^c{}_\nu \partial_a e^\mu{}_c,$$

using our standard trick of considering $0 = \partial_b(e^\mu{}_a e^a{}_\nu)$. We now compute the RHS. We have:

$$\begin{gathered}(d\omega^\mu{}_\nu)_{ab} + 2\omega^\mu{}_{\rho[a|}\omega^\rho{}_{\nu|b]} = \\ 2\partial_{[a|}\left(e^\mu{}_d e^c{}_\nu \Gamma^g{}_{|b]c} - e^c{}_\nu \partial_{|b]} e^\mu{}_c\right) + \\ 2e^c{}_\sigma \left(\Gamma^d_{[a|c} e^\mu{}_d - \partial_{[a|} e^\mu{}_c\right) e^f{}_\nu \left(\Gamma^g_{|b]f} e^\sigma{}_g - \partial_{|b]} e^\sigma{}_f\right)\end{gathered}$$

Expanding all the brackets, we get a large amount of cancellation (we need to use $0 = \partial_b(e^\mu{}_a e^a{}_\nu)$), leading to

$$\begin{aligned}(d\omega^\mu{}_\nu)_{ab} + 2\omega^\mu{}_{\rho[a|}\omega^\rho{}_{\nu|b]} &= 2e^\mu{}_d e^c{}_\nu \left(\partial_{[a}\Gamma^d_{|b]c} + \Gamma^d_{[a|f}\Gamma^f_{|b]c}\right) \\ &= e^\mu{}_d e^c{}_\nu R^d{}_{cab}.\end{aligned}$$

We worked out the ab component, which sits next to $dx^a \wedge dx^b = e^a{}_\rho e^b{}_\sigma dE^\rho \wedge dE^\sigma$. This gives us the answer when we recall that a p-form A has components $A = \frac{1}{p!} A_{a_1 \ldots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p}$ (this in particular gives the factor of $1/2$). □

In particular, the components of the Riemann tensor computed using the Christoffel symbols and spacetime indices are the same as those computed using the curvature tensor and the frame field indices.

13.5 Example computation

To highlight the utility of this, let's consider the following example:

Example: Consider the metric

$$ds^2 = -W(r)^2 dt^2 + \frac{dr^2}{W(r)^2} + r^2(d\theta^2 + \sin^2(\theta) d\phi^2).$$

Suppose we want to construct the Riemann tensor for this metric. We can do so using vierbein fields as follows.

First, we construct a basis of one-forms as follows: $E^0 = W dt$, $E^1 = dr/W$, $E^2 = r d\theta$ and $E^3 = r\sin(\theta) d\phi$. Inverting,

$$dt = \frac{E^0}{W}, \quad dr = W E^1, \quad d\theta = \frac{E^2}{r}, \quad d\phi = \frac{E^3}{r\sin(\theta)}.$$

Therefore the metric becomes:

$$ds^2 = -(E^0)^2 + (E^1)^2 + (E^2)^2 + (E^3)^2.$$

We want to compute the curvature 2-form, $\Omega^\mu{}_\nu$, so on the way we need to compute $\omega^\mu{}_\nu$ and therefore need dE^μ. Calculating, we have

$$\begin{gathered}dE^0 = d(W dt) = dW \wedge dt + W \wedge d(dt) = W'(r) dr \wedge dt, \\ dE^1 = d\left(\frac{dr}{W}\right) = d\left(\frac{1}{W}\right) \wedge dr = -\frac{W'(r)}{W(r)^2} dr \wedge dr = 0, \\ dE^2 = dr \wedge d\theta, \\ dE^3 = \sin(\theta) dr \wedge d\phi + r\cos(\theta) d\theta \wedge d\phi.\end{gathered}$$

Rewriting in terms of the one-forms, we have:

$$\begin{gathered}dE^0 = -W' E^0 \wedge E^1, \\ dE^1 = 0, \\ dE^2 = \frac{W}{r} E^1 \wedge E^2, \\ dE^3 = \frac{W}{r} E^1 \wedge E^3 + \frac{\cot(\theta)}{r} E^2 \wedge E^3.\end{gathered}$$

Now use Cartan's first equation of structure:

$$dE^\mu = -\omega^\mu{}_\nu \wedge E^\nu.$$

This implies that

$$dE^0 = -\omega^0{}_1 \wedge E^1 - \omega^0{}_2 \wedge E^2 - \omega^0{}_3 \wedge E^3.$$

Recalling that $dE^0 = -W' E^0 \wedge E^1$, we see that the RHS ω values are very restricted. In particular, $\omega^0{}_1 \sim aE^0 + bE^1$, $\omega^0{}_2 \sim E^2$, $\omega^0{}_3 \sim E^3$, else we wouldn't get the desired dE^0 value.

Similarly,

$$dE^1 = -\omega^1{}_0 \wedge E^0 - \omega^1{}_2 \wedge E^2 - \omega^1{}_3 \wedge E^3.$$

Now note that $\omega^1{}_0 = \omega_{10} = -\omega_{01} = \omega^0{}_1$ (since we can raise and lower indices like an object in special relativity, and the connection one-form has antisymmetric components by definition). Thus $\omega^1{}_0 = \omega^0{}_1$. We also see from the form of dE^1 that $\omega^1{}_0$ must look like E^0. Thus we deduce that

$$\omega^0{}_1 = W'E^0.$$

Similarly, $\omega^0{}_2 = 0$, $\omega^0{}_3 = 0$. We can keep going with the other values to get:

$$\omega^1{}_2 = -\frac{W}{r}E^2, \quad \omega^1{}_3 = -\frac{W}{r}E^3, \quad \omega^2{}_3 = -\frac{\cot(\theta)}{r}E^3.$$

The final step is to compute the curvature 2-form, which is given by Cartan's second equation of structure as:

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu.$$

Computing each component in turn, we get:

$$\begin{aligned}\Omega^0{}_1 &= -(W''W + (W')^2)E^0 \wedge E^1,\\ \Omega^0{}_2 &= -\frac{WW'}{r}E^0 \wedge E^2,\\ \Omega^1{}_2 &= -\frac{WW'}{r}E^1 \wedge E^2,\\ \Omega^1{}_3 &= -\frac{WW'}{r}E^1 \wedge E^3,\\ \Omega^2{}_3 &= \left(\frac{1}{r^2} - \frac{W^2}{r^2}\right) E^2 \wedge E^3.\end{aligned}$$

Since $\Omega^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\rho\sigma}E^\rho \wedge E^\sigma$, this now contains all the information about the Riemann tensor.

In particular, we can calculate Ricci tensor components easily:

$$R_{00} = R^\mu{}_{0\mu 0} = R^1{}_{010} + R^2{}_{020} + R^3{}_{030}.$$

From the curvature 2-form, we see that

$$R^0{}_{101} = -WW'' - (W')^2, \quad R^0{}_{303} + R^0{}_{202} = -\frac{2WW'}{r}.$$

and so $R_{00} = WW'' + (W')^2 + 2WW'/r$. We can also deduce:

$$R_{11} = -WW' - (W')^2 - \frac{2WW'}{r}, \quad \text{and}$$
$$R_{22} = R_{33} = -\frac{2WW'}{r} - \frac{W^2}{r^2} + \frac{1}{r^2}.$$

This is a lot easier than calculating all the Christoffel symbols!

13.6 The Palatini action

Definition: The *Palatini action* is given by

$$I = \int \eta_{\eta\nu\rho\sigma}\left(\Omega^{\mu\nu} \wedge E^\rho \wedge E^\sigma\right),$$

where η is the alternating symbol (not tensor!), Ω is the curvature 2-form, and E^μ are the basis one-forms.

Theorem: Treating $\omega^\mu{}_\nu$ and E^μ as independent variables, the Palatini action has the vacuum Einstein equations as its equations of motion.

Proof: Let $\omega^\mu{}_\nu \mapsto \omega^\mu{}_\nu + \delta\omega^\mu{}_\nu$ and $E^\mu \mapsto E^\mu + \delta E^\mu$. Then by Cartan's second equation of structure, we have

$$\begin{aligned}\eta_{\mu\nu\rho\sigma}\delta\Omega^\mu{}_\nu &= \eta_{\mu\nu\rho\sigma}\left(d(\delta\omega^\mu{}_\nu) + \delta\omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu + \omega^\mu{}_\alpha \wedge \delta\omega^\alpha{}_\nu\right)\\ &= \eta_{\mu\nu\rho\sigma}\left(d(\delta\omega^\mu{}_\nu) + 2\delta\omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu\right).\end{aligned}$$

We also have:

$$\eta_{\mu\nu\rho\sigma}\delta(E^\rho \wedge E^\sigma) = 2\eta_{\mu\nu\rho\sigma}E^\rho \wedge \delta E^\sigma.$$

Hence calculating the variation of the action, and integrating by parts, we have: $\delta I =$

$$= \int \eta_{\mu\nu\rho\sigma}\bigg(- \delta\omega^{\mu\nu} \wedge d(E^\rho \wedge E^\sigma) + 2\delta\omega^\mu{}_\alpha \wedge \omega^{\alpha\nu} \wedge E^\rho \wedge E^\sigma$$
$$+2\Omega^{\mu\nu} \wedge E^\rho \wedge \delta E^\sigma\bigg).$$

Therefore, the equations of motion are:

$$\begin{aligned}0 &= \eta_{\mu\nu\rho\sigma}\left(-\eta^{\alpha\nu}d(E^\rho \wedge E^\sigma) + 2\omega^{\alpha\nu} \wedge E^\rho \wedge E^\sigma\right),\\ 0 &= \eta_{\mu\nu\rho\sigma}\Omega^{\mu\nu} \wedge E^\rho.\end{aligned}$$

1ST EQUATION: We note that $\eta_{\mu\nu\rho\sigma}d(E^\rho \wedge E^\sigma) = 2\eta_{\mu\nu\rho\sigma}dE^\rho \wedge E^\sigma = 2\eta_{\mu\nu\rho\sigma}(\omega^\rho{}_\beta \wedge E^\beta - \Theta^\rho) \wedge E^\sigma$, by Cartan's first equation of structure. Thus the equation becomes:

$$0 = \eta_{\mu\nu\rho\sigma}\left(-2\eta^{\alpha\nu}(\omega^\rho{}_\beta \wedge E^\beta - \Theta^\rho) \wedge E^\sigma + 2\omega^{\alpha\nu} \wedge E^\rho \wedge E^\sigma\right).$$

Lowering the α, this is equivalent to:

$$0 = 2\eta_{\mu\nu\rho\sigma}\left(-\delta^\nu{}_\alpha(\omega^\rho{}_\beta \wedge E^\beta - \Theta^\rho) + 2\omega_\alpha{}^\nu \wedge E^\rho\right) \wedge E^\sigma$$

Wedging both sides with E^α, we're left with $0 = 2\eta_{\mu\alpha\rho\sigma}\Theta^\rho \wedge E^\sigma \wedge E^\alpha$. That is, this equation just encodes the fact the torsion vanishes.

2ND EQUATION: Writing out equation two using the definition of the curvature 2-form we have:

$$\frac{1}{2}R^{ab}{}_{cd}\eta_{\mu\nu\rho\sigma}e^\mu{}_a e^\nu{}_b e^\rho{}_\sigma dx^c \wedge dx^d \wedge dx^e = 0.$$

Wedge with dx^f, and use $dx^c \wedge dx^d \wedge dx^e \wedge dx^f \propto \eta^{cdef}\epsilon$, where ϵ is the alternating tensor. Finally, use the fact that

$$\eta_{abeg}\eta^{cdef} = -\delta^{abg}_{cdf} = -6\delta^{[a}_c\delta^b_d\delta^{g]}_f,$$

from far earlier in the course. Substituting in gives the vacuum Einstein equations. □