Part III: Quantum Field Theory - Revision

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1 Classical field theory

1.1 Lagrangian dynamics of fields

Definition: The *Lagrangian* of a system of fields $\phi_a(x)$, $x \in \mathbb{R}^4$, is a function $L \equiv L(\phi_a, \partial_\mu \phi_a)$. A *Lagrangian density* for L is a function $\mathcal{L}$ obeying:

$$L = \int d^3\mathbf{x}\ \mathcal{L}(\phi_a, \partial_\mu \phi_a).$$

Note the Lagrangian density is not unique; we can add on any three-divergence $\nabla \cdot \mathbf{A}$ and get the same Lagrangian.

Definition: The *action* of a system of fields is

$$S = \int d^4x\ \mathcal{L}(\phi_a, \partial_\mu \phi_a),$$

i.e. it is the time integral of the Lagrangian.

Least Action Principle: Fields evolve such that S is stationary with respect to field variations which have fixed initial and final values.

Theorem: The dynamics of fields are given by the *Euler-Lagrange equations*:

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0.$$

Proof: Let $\phi_a \mapsto \phi_a + \delta\phi_a$. Then S transforms to:

$$S \mapsto S + \sum_a \int d^4x\ \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right) \delta\phi_a,$$

using integration by parts (and recalling fields vanish at spatial infinity, and have fixed initial and final values so that $\delta\phi(t_{\text{init}}, \mathbf{x}) = \delta\phi(t_{\text{fin}}, \mathbf{x}) = 0$). The result follows. □

Example: The *Klein-Gordon* Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The equation of motion is $\partial_\mu\partial^\mu\phi + m^2\phi = 0$, called the *Klein-Gordon equation*.

Trialling a wavelike solution $\phi(x) = e^{-ip\cdot x}$, with $x = (t, \mathbf{x})$, $p = (E, \mathbf{p})$, the equation implies $E^2 = |\mathbf{p}|^2 + m^2$, i.e. the relativistic energy equation for a particle of mass m.

Example: Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2.$$

The minus sign is necessary to ensure the kinetic terms are positive. The equation of motion is $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$, which if we define the *field-strength tensor* $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ can be written compactly as $\partial_\mu F^{\mu\nu} = 0$.

Definition: A *local Lagrangian* has no terms coupling $\phi(t, \mathbf{x})$ and $\phi(t, \mathbf{y})$ for $\mathbf{x} \neq \mathbf{y}$.

In this course we only use local Lagrangians.

1.2 Lorentz invariance

Definition: A *Lorentz transformation* is a matrix Λ satisfying $\Lambda^T\eta\Lambda = \eta$ (where η is the Minkowski metric). In index notation, this is $\Lambda^\sigma{}_\mu \eta_{\sigma\tau} \Lambda^\tau{}_\nu = \eta_{\mu\nu}$, or (taking the inverse):

$$\Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}.$$

Definition: Under a *passive* Lorentz transformation, a scalar field $\phi(x)$ transforms as $\phi(x) \mapsto \phi'(x) = \phi(\Lambda x)$ (i.e. just a relabelling of coordinates). Under an *active* Lorentz transformation, the field transforms as $\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1}x)$ (i.e. field itself moves position).

In this course, we use only active transformations (but passive transformations are completely equivalent since the inverse of a Lorentz transformation is a Lorentz transformation).

Definition: If the action S of a theory is invariant under Lorentz transformations, we say the theory is *Lorentz invariant*.

Theorem: Any theory with Lagrangian density $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$ is Lorentz invariant.

Proof: Let $x' = \Lambda^{-1}x$. The potential density $U(\phi)$ transforms as:

$$U(x) \equiv U(\phi(x)) \mapsto U(\phi'(x)) = U(\phi(x')) \equiv U(x'),$$

so U transforms as a scalar field.

The derivative $\partial_\mu \phi(x)$ transforms as:

$$\partial_\mu \phi(x) \mapsto \partial_\mu \phi(x') = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial \phi(x')}{\partial x'^\nu} = (\Lambda^{-1})^\nu{}_\mu \partial'_\nu \phi(x').$$

Using this we see derivative term transforms as:

$$\partial_\mu \phi(x) \partial^\mu \phi(x) \mapsto \partial'_\nu \phi(x') \partial'^\nu \phi(x'),$$

using $\Lambda^T \eta \Lambda = \eta$. Hence

$$\mathcal{L}(\phi(x), \partial_\mu \phi(x)) \equiv \mathcal{L}(x) \mapsto \mathcal{L}(x'),$$

under a Lorentz transformation, and hence $\mathcal{L}$ is a Lorentz scalar. So the action transforms as:

$$S \mapsto S' = \int d^4x \, \mathcal{L}(x') = \int d^4x \, \mathcal{L}(\Lambda^{-1}x).$$

Now change variables as $y = \Lambda^{-1}x$. The Jacobian is $\det(\Lambda^{-1}) = 1$ for Λ in the special Lorentz group, and so the action is invariant. □

The above Theorem also shows that vector fields transform as $A_\mu(x) \mapsto A'_\mu(x) = (\Lambda^{-1})^\nu{}_\mu A_\nu(\Lambda^{-1}x)$.

1.3 Noether's theorem

Noether's Theorem: Every continuous symmetry of a field theory gives rise to a *conserved current* j^μ obeying $\partial_\mu j^\mu = 0$.

Proof: Let $\phi \mapsto \phi + \delta\phi$ be a symmetry. Then the Lagrangian must be invariant up to a four-divergence (so that the action is invariant): $\mathcal{L} \mapsto \mathcal{L} + \partial_\mu X^\mu$. Taylor expanding the transformed $\mathcal{L}$, we also have:

$$\begin{aligned}\mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \partial_\mu \delta\phi) &= \mathcal{L} + \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + (\partial_\mu \delta\phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \\ &= \mathcal{L} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right),\end{aligned}$$

using the Euler-Lagrange equations. Comparing both expressions, we must have:

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - X^\mu \right)}_{j^\mu} = 0. \quad \square$$

Theorem: A conserved current j^μ satisfying $j^i \to 0$ as $|\mathbf{x}| \to \infty$ gives rise to a conserved charge

$$Q = \int_{\mathbb{R}^3} d^3\mathbf{x} \, j^0.$$

Proof: We have:

$$\frac{dQ}{dt} = \int d^3\mathbf{x} \, \partial_0 j^0 = -\int d^3\mathbf{x} \, \nabla \cdot \mathbf{j} = 0,$$

by $\partial_\mu j^\mu = 0$ and the divergence theorem. □

Example: Suppose $\mathcal{L}$ does not depend on x explicitly. Consider an infinitesimal translation $x^\nu \mapsto x^\nu - \epsilon^\nu$. The fields transform as:

$$\phi(x) \mapsto \phi(x^\nu + \epsilon^\nu) = \phi(x) + \epsilon^\nu \partial_\nu \phi(x).$$

Also since the Lagrangian is a scalar,

$$\mathcal{L}(x) \mapsto \mathcal{L}(x^\nu + \epsilon^\nu) = \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}(x) = \mathcal{L}(x) + \epsilon^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L}(x)).$$

Following Noether's Theorem, we see we get one conserved quantity for each component of ϵ^ν:

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}.$$

Definition: We call $T^\mu{}_\nu = (j^\mu)_\nu$ (in the above) the *energy-momentum tensor*. The associated conserved charges are the *total energy* and the *total momentum*:

$$E = \int d^3\mathbf{x} \, T^{00}, \quad P^i = \int d^3\mathbf{x} \, T^{0i}.$$

Theorem: If $T^{\mu\nu}$ is not symmetric, we can make it symmetric. Let $\Gamma^{\rho\mu\nu}$ be antisymmetric on the first two indices. Then $T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu}$ is a symmetric conserved quantity for some choice of Γ.

Proof: $\partial_\mu (T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu}) = (\partial_\mu \partial_\rho) \Gamma^{\rho\mu\nu} = 0$, since we have antisymmetric indices on Γ and symmetric indices on the derivatives. So conserved.

Choose Γ to obey $\partial_\rho (\Gamma^{\rho\mu\nu} - \Gamma^{\rho\nu\mu}) = 0$ for symmetry on μ, ν indices. □

1.4 Hamiltonian dynamics of fields

Definition: The *conjugate momentum* of the field ϕ is defined to be:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}.$$

Definition: The *Hamiltonian density* is $\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$, where $\dot{\phi}$ is eliminated everywhere for π. The *Hamiltonian* is:

$$H = \int d^3\mathbf{x} \, \mathcal{H}.$$

Note: In the Hamiltonian formalism, there are also equations for dynamics (*Hamilton's equations*). These are given by:

$$\dot{\phi} = \frac{\partial H}{\partial \pi}, \quad \dot{\pi} = -\frac{\partial H}{\partial \phi},$$

where the derivatives are *functional derivatives*.

Also note that the Hamiltonian formalism is Lorentz invariant because it is equivalent to the Lagrangian formalism.

1.5 Angular momentum of classical fields

Example: Consider a Lorentz transformation $\Lambda^\mu{}_\nu$. The infinitesimal generator of this transformation, given by

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\omega^\mu{}_\nu,$$

can easily by shown to be antisymmetric (i.e. $\omega_{\mu\nu} = -\omega_{\nu\mu}$) using $\Lambda^T\eta\Lambda = \eta$.

Choosing specific ω's gives rotations (choosing ω to have only non-zero entries in lower right 3×3 block) and boosts (choosing only non-zero entries to be in rest of matrix).

A Lorentz-invariant theory gives us conserved quantities from $\omega^\mu{}_\nu$.

Theorem: A scalar field transforms as

$$\phi'(x) = \phi(x) - \epsilon\omega^\mu{}_\nu x^\nu \partial_\mu\phi(x),$$

under the above infinitesimal Lorentz transformation.

Proof: We know $\phi'(x) = \phi(\Lambda^{-1}x)$. So just need Λ^{-1}. But $\Lambda^T\eta\Lambda = \eta$ implies $(\Lambda^{-1})^\mu{}_\nu = \eta^{\mu\sigma}\Lambda^\tau{}_\sigma\eta_{\tau\nu}$. Hence $(\Lambda^{-1})^\mu{}_\nu = \delta^\mu{}_\nu - \epsilon\omega^\mu{}_\nu$. Substituting into $\phi(\Lambda^{-1}x)$ and Taylor expanding, we get the result. □

Since the Lagrangian is a scalar field, the Lagrangian also transforms as:

$$\mathcal{L}'(x) = \mathcal{L}(x) - \epsilon\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}(x).$$

Notice that $\partial_\mu(\omega^\mu{}_\nu\mathcal{L}) = \omega^\mu{}_\nu x^\nu\partial_\mu\mathcal{L}(x)$, by a short calculation. So the Lagrangian changes by a total derivative!

It follows from Noether's Theorem that we have a conserved current:

$$j^\mu = -\omega^\rho{}_\nu x^\nu T^\mu{}_\rho,$$

after a short calculation. The associated conserved charge is

$$Q = -\int d^3x\ \omega_{\rho\nu}T^{0\rho}x^\nu.$$

For ω a rotation generator, only Latin indices survive. Indeed, we can write $\omega_{jk} = \epsilon_{ijk}\phi^i$ in the standard way. Then the conserved charge is:

$$Q = -\epsilon_{ijk}\int d^3x\ \phi^i T^{0j}x^k = \frac{1}{2}\epsilon_{ijk}\int d^3x\ \phi^i\left(T^{0k}x^j - T^{0j}x^k\right).$$

Taking $\phi^i = (1,0,0),(0,1,0),(0,0,1)$ in succession gives the three conserved quantities:

Definition: The *conserved angular momentum* of a field is:

$$Q_i = \frac{1}{2}\epsilon_{ijk}\int d^3x\ \left(T^{0k}x^j - T^{0j}x^k\right).$$

2 Free quantum field theory

2.1 Second quantisation

Definition: To quantise a system of fields, we use the *second quantisation scheme*. We promote the fields $\phi_a(\mathbf{x})$, $\pi_b(\mathbf{x})$ (as functions of three-position) to operators labelled by $\mathbf{x}$ (that is, there are two operators for each 3-position). We impose the commutation relations:

$$[\pi_a(\mathbf{x}),\pi_b(\mathbf{y})] = 0, \quad [\phi_a(\mathbf{x}),\phi_b(\mathbf{y})] = 0,$$
$$[\phi_a(\mathbf{x}),\pi^b(\mathbf{y})] = i\delta_a{}^b\delta^3(\mathbf{x}-\mathbf{y}).$$

An operator-valued function of space is a *quantum field*.

2.2 Quantising Klein-Gordon theory

Theorem: The Hamiltonian of Klein-Gordon theory is

$$H = \int d^3\mathbf{x}\ \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right).$$

Proof: Quick calculation. □

We now quantise, so promote ϕ and π to quantum fields obeying the above commutation relations.

To solve the theory, we use the following trick. Write ϕ and π in terms of *creation* and *annihilation operators* in momentum space (analogous to quantum harmonic oscillator):

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(a_\mathbf{p}e^{i\mathbf{x}\cdot\mathbf{p}} + a_\mathbf{p}^\dagger e^{-i\mathbf{x}\cdot\mathbf{p}}\right),$$
$$\pi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}\left(a_\mathbf{p}e^{i\mathbf{x}\cdot\mathbf{p}} - a_\mathbf{p}^\dagger e^{-i\mathbf{x}\cdot\mathbf{p}}\right),$$

where $\omega_\mathbf{p} = \sqrt{|\mathbf{p}|^2+m^2} = E_\mathbf{p}$. Inverting, we find the definitions of the operators $a_\mathbf{p}$ and $a_\mathbf{p}^\dagger$:

$$a_\mathbf{p} = \int d^3\mathbf{x}\left(\sqrt{\frac{\omega_\mathbf{p}}{2}}\phi(\mathbf{x}) + \frac{i}{\sqrt{2\omega_\mathbf{p}}}\pi(\mathbf{x})\right)e^{-i\mathbf{x}\cdot\mathbf{p}},$$
$$a_\mathbf{p}^\dagger = \int d^3\mathbf{x}\left(\sqrt{\frac{\omega_\mathbf{p}}{2}}\phi(\mathbf{x}) - \frac{i}{\sqrt{2\omega_\mathbf{p}}}\pi(\mathbf{x})\right)e^{i\mathbf{x}\cdot\mathbf{p}}.$$

To obtain this, we take the Fourier transform of $\phi(\mathbf{x})$, $\pi(\mathbf{x})$ and use the delta function identity:

$$\delta^3(\mathbf{x}) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{x}\cdot\mathbf{p}}.$$

Theorem: $[a_\mathbf{p}, a_\mathbf{q}] = 0$, $[a_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger] = 0$, $[a_\mathbf{p}, a_\mathbf{q}^\dagger] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$.

Proof: Follows directly from definitions of $a_\mathbf{p}$, $a_\mathbf{p}^\dagger$. □

Theorem: The Hamiltonian of quantum Klein-Gordon theory may be written as:

$$H = \frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_{\mathbf{p}}\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right).$$

Proof: Substitute in the expressions for $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ in terms of the creation and annihilation operators, and be very careful with the algebra! □

2.3 The free vacuum and normal ordering

Definition: The *vacuum* state $|0\rangle$ is the state for which $a_{\mathbf{p}}|0\rangle = 0$.

Theorem: The energy of the vacuum state is infinite: $H|0\rangle = \infty|0\rangle$.

Proof: We have:

$$H|0\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2}\omega_{\mathbf{p}}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]|0\rangle = \frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_{\mathbf{p}}\delta^3(\mathbf{0})|0\rangle. \square$$

There are two kinds of divergence here. The first is because space is infinitely large (called *infrared divergence*); if we worked in a finite volume V instead, we would have:

$$(2\pi)^3\delta^3(\mathbf{0}) = \int d^3\mathbf{x}\, e^{i\mathbf{x}\cdot\mathbf{0}} = V.$$

So we can safely remove the delta function by considering *energy density* instead. Then we get a second divergence (called *ultraviolet divergence*) from:

$$\frac{E}{V} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2}\omega_{\mathbf{p}}.$$

This is because $\mathbf{p}$ can be arbitrarily large; i.e. we're assuming the theory holds for arbitrarily small distance scales. Instead, we should cut off the integral at some Λ, called the *ultraviolet cutoff*, where the theory breaks down.

To fix the problem of infinite energy practically, we introduce *normal ordering* of operators.

Definition: Let $\phi_1(\mathbf{x}_1)...\phi_n(\mathbf{x}_n)$ be a string of operators. Its *normal ordering* is the same string, but with all creation operators moved to the left and all annihilation operators moved to the right. The normal ordered string is written:

$$:\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)...\phi_n(\mathbf{x}_n):$$

In particular, the normal-ordered Hamiltonian is:

$$:H:= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}},$$

and so $H|0\rangle = 0$, i.e. we've removed the infinite constant.

2.4 Momentum in the quantum theory

Theorem: The quantum normal-ordered momentum, $\mathbf{P}$, in Klein-Gordon theory is:

$$\mathbf{P} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathbf{p}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}.$$

Proof: Computing the energy-momentum tensor, we see that $T^{0i} = -\pi\nabla\phi$. Hence:

$$\mathbf{P} = -\int d^3\mathbf{x}\,\pi\nabla\phi = ... = \frac{1}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right),$$

after a calculation (using the fact odd terms integrate to zero). After normal ordering, we get the required expression. □

2.5 Particles in Klein-Gordon theory

From now on, redefine H to be the normal ordered Hamiltonian. Then:

Theorem: We have $[H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}}a_{\mathbf{p}}$, $[H, a_{\mathbf{p}}^\dagger] = \omega_{\mathbf{p}}a_{\mathbf{p}}^\dagger$.

Proof: Quick calculation using relation $[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$. □

Definition: We define the state $|\mathbf{p}\rangle$ by $|\mathbf{p}\rangle = a_{\mathbf{p}}^\dagger|0\rangle$.

Theorem: (i) $H|\mathbf{p}\rangle = \omega_{\mathbf{p}}|\mathbf{p}\rangle$; (ii) $\mathbf{P}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$, where $\mathbf{P}$ is the normal-ordered momentum operator.

Proof: (i) $H|\mathbf{p}\rangle = Ha_{\mathbf{p}}^\dagger|0\rangle = [H, a_{\mathbf{p}}^\dagger]|0\rangle = \omega_{\mathbf{p}}|\mathbf{p}\rangle$. For (ii), we have:

$$\mathbf{P}|\mathbf{p}\rangle = \int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\mathbf{q}a_{\mathbf{q}}^\dagger[a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger]|0\rangle = \mathbf{p}|\mathbf{p}\rangle \quad \square$$

We can interpret the results of the above Theorem as follows. (i) The state has energy $E = \sqrt{|\mathbf{p}|^2 + m^2}$; (ii) the state is a momentum eigenstate with momentum $\mathbf{p}$. This shows that the state $|\mathbf{p}\rangle$ is a particle of momentum $\mathbf{p}$ and mass m.

2.6 Multi-particle states

Definition: We define the *multi-particle state* $|\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n\rangle$ to be $a_{\mathbf{p}_1}^\dagger...a_{\mathbf{p}_n}^\dagger|0\rangle$.

Note that since the $a_{\mathbf{p}}^\dagger$ commute, multi-particle states are symmetric under exchange of particles in this theory. So these particles are *bosons*.

Definition: The space spanned by the multi-particle states is called *Fock space*.

2.7 Relativistic normalisation

Let the vacuum be normalised as $\langle 0|0\rangle = 1$. Then

$$\langle \mathbf{p}|\mathbf{q}\rangle = \langle 0|a_{\mathbf{p}}a_{\mathbf{q}}^{\dagger}|0\rangle = \langle 0|[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}]|0\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).$$

Note $\langle \mathbf{p}|\mathbf{q}\rangle$ is a scalar, so should be Lorentz invariant. But it isn't! To fix this, we need to normalise our states differently.

Theorem: (i) The measure

$$\int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}}$$

is Lorentz invariant; (ii) $2E_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{q})$ is Lorentz invariant.

Proof: (i) Notice $\int d^4p$ is trivially Lorentz invariant. Also note the dispersion relation $p_0^2 = |\mathbf{p}|^2 + m^2$ is trivially Lorentz invariant. Hence

$$\int d^4p\, \delta(p_0^2 - |\mathbf{p}|^2 - m^2)\Big|_{p_0>0}$$

is a Lorentz invariant. Using the identity

$$\delta(g(x)) = \sum_{x_i \text{ roots of } g} \frac{\delta(x-x_i)}{|g'(x_i)|},$$

we're done. (ii) follows since

$$\int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}}\, 2E_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{q}) = 1.$$

The measure is Lorentz invariant, and 1 is Lorentz invariant, so $2E_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{q})$ must be Lorentz invariant too. □

This Theorem shows that if we define $|p\rangle = \sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle$, we get

$$\langle p|q\rangle = \sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{q}}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}),$$

and so we're saved.

Definition: We define the *relativistically normalised* momentum eigenstates to be $|p\rangle = \sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle$. Throughout the rest of the course we will use relativistically normalised states.

2.8 Complex scalar field theory

The quantisation of Klein-Gordon theory generalises to complex scalar fields easily.

Definition: *Free complex scalar field theory* is described by the Lagrangian

$$\mathcal{L} = \partial_\mu\psi\partial^\mu\psi^* - \mu^2\psi^*\psi.$$

To solve quantum complex scalar field theory, we introduce raising and lowering operators as before. But since ψ is complex, $\psi^\dagger = \psi^*$ is not necessarily equal to ψ, so we get some extra creation and annihilation operators:

$$\psi = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{i\mathbf{x}\cdot\mathbf{p}} + c_{\mathbf{p}}^{\dagger}e^{-i\mathbf{x}\cdot\mathbf{p}}\right)$$

$$\pi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} i\sqrt{\frac{E_{\mathbf{p}}}{2}}\left(b_{\mathbf{p}}^{\dagger}e^{-i\mathbf{x}\cdot\mathbf{p}} - c_{\mathbf{p}}e^{i\mathbf{x}\cdot\mathbf{p}}\right)$$

This time π comes from $\pi = \dot{\psi}^* = \partial\mathcal{L}/\partial\dot{\psi}$, i.e. we get a change in sign of the i due to the complex conjugate.

The commutation relations are:

$$[\psi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}), \quad [\psi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})] = -i\delta^3(\mathbf{x}-\mathbf{y}),$$

with all others zero. These imply the commutation relations:

Theorem: The creation and annihilation operators obey: $[b_{\mathbf{p}}, b_{\mathbf{q}}^{\dagger}] = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})$ and $[c_{\mathbf{p}}, c_{\mathbf{q}}^{\dagger}] = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})$, with all other commutation relations zero.

Proof: Same as in real scalar case. □

We can also compute the normal-ordered Hamiltonian:

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, E_{\mathbf{p}}(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + c_{\mathbf{p}}^{\dagger}c_{\mathbf{p}}).$$

2.9 Conserved charges

In both Klein-Gordon theory and in free complex scalar field theory we can construct a special conserved charge Q which we associate with particle number.

In Klein-Gordon theory, we introduce (out of thin air):

Definition: The *number operator* is given by:

$$N = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}.$$

This is called the number operator because it counts the number of particles in a multi-particle state:

Theorem: $N\,|\mathbf{p}_1, ..., \mathbf{p}_n\rangle = n\,|\mathbf{p}_1, ..., \mathbf{p}_n\rangle$.

Proof: We have

$$N\,|\mathbf{p}_1, ..., \mathbf{p}_n\rangle = \int \frac{d^3\mathbf{q}}{(2\pi)^3} a_{\mathbf{q}}^{\dagger}a_{\mathbf{q}}a_{\mathbf{p}_1}^{\dagger}...a_{\mathbf{p}_n}^{\dagger}\,|0\rangle.$$

Now commute $a_{\mathbf{q}}$ past each $a_{\mathbf{p}_i}^{\dagger}$, and use induction. □

Theorem: $[N, H] = 0$, i.e. particle number is conserved.

Proof: Tedious calculation. □

Constructing a number operator is easier in complex scalar field theory, since we can use *internal symmetry* of the Lagrangian:

Definition: An *internal symmetry* of a theory is a symmetry of the Lagrangian involving only a transformation of the fields.

In complex scalar field theory, we have that $\psi \mapsto e^{i\alpha}\psi$ is an internal symmetry of the Lagrangian. By Noether's Theorem, this gives rise to the conserved quantity:

$$Q = i\int d^3\mathbf{x}\,(\dot{\psi}^*\psi - \psi^*\dot{\psi}) = i\int d^3\mathbf{x}\,(\pi\psi - \psi^\dagger\pi^\dagger).$$

In the quantum theory, this is the normal-ordered operator:

$$Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3}(c^\dagger_{\mathbf{p}}c_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}),$$

i.e. it consists of the difference of two number operators N_b and N_c of Klein-Gordon form.

Theorem: $[Q,H] = 0$, i.e. the difference between N_b and N_c is conserved.

Proof: Tedious calculation. □

The fact Q is conserved is important. It gives the interpretation that b and c correspond to two different particles, whose difference is always constant. Thus we should view b and c particles as particles and corresponding *antiparticles* (of opposite charge).

2.10 The Heisenberg picture

Definition: *Heisenberg operators* are defined from Schrödinger operators by: $O_H(t) = e^{iHt}O_S(t)e^{-iHt}$. *Heisenberg states* are defined by $|\psi\rangle_H = |\psi(0)\rangle_S$ (i.e. the corresponding Schrödinger state at time 0).

Clearly, multiplying by e^{iHt} on the left and e^{-iHt} on the right, we arrive at *equal time commutation relations* for the quantum fields in the Heisenberg picture:

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}),$$

with all others zero. The Heisenberg picture also gives an equation of motion for the quantum fields:

Theorem (Heisenberg's equation of motion):
We have: $\dot{O}_H(t) = i[H, O_H]$.

Proof: Use the Definition of the Heisenberg operators. □

Theorem: $e^{iHt}a_{\mathbf{p}}e^{-iHt} = e^{-iE_{\mathbf{p}}t}a_{\mathbf{p}}$, $e^{iHt}a^\dagger_{\mathbf{p}}e^{-iHt} = e^{iE_{\mathbf{p}}t}a^\dagger_{\mathbf{p}}$.

Proof: Use Maclaurin expansion of LHS, together with commutators $[H,a_{\mathbf{p}}] = -E_{\mathbf{p}}a_{\mathbf{p}}$, and $[H,a^\dagger_{\mathbf{p}}] = E_{\mathbf{p}}a^\dagger_{\mathbf{p}}$. □

This immediately gives the Heisenberg quantum fields:

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ix\cdot p} + a^\dagger_{\mathbf{p}}e^{ix\cdot p}\right),$$

$$\pi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{E_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{-ix\cdot p} - a^\dagger_{\mathbf{p}}e^{ix\cdot p}\right).$$

These expansions are also now manifestly Lorentz invariant. Note also: the Heisenberg Hamiltonian is the same as the Schrödinger Hamiltonian.

It's also possible to recover the Klein-Gordon equation in *operator* form from the Heisenberg picture:

$$\begin{aligned}\dot{\phi} &= i[H,\phi(x)]\\ &= \frac{i}{2}\int d^3\mathbf{y}\,([\pi(y)^2,\phi(x)] + \nabla_y\phi(y)\nabla_y[\phi(y),\phi(x)]\\ &\quad+\nabla_y[\phi(y),\phi(x)]\nabla_y\phi(y)) = \ldots = \pi(x).\end{aligned}$$

Similarly, $\dot{\pi} = \nabla^2\phi - m^2\phi$. Combining these equations gives the Klein-Gordon equation.

2.11 Quantised angular momentum

Now we have the Heisenberg picture, it is easy to show that particles in Klein-Gordon theory have spin 0.

Theorem: The normal-ordered quantum angular momentum operator in KG theory is:

$$J_i = -\frac{i}{2}\epsilon_{ijk}\int \frac{d^3\mathbf{p}}{(2\pi)^3}a^\dagger_{\mathbf{p}}\left(p^j\frac{\partial}{\partial p_k} - p^k\frac{\partial}{\partial p_j}\right)a_{\mathbf{p}}.$$

Proof: Putting the KG energy-momentum tensor into the general angular momentum conserved charge from earlier, we find:

$$J_i = \epsilon_{ijk}\int d^3x\,\pi(x)x^j\partial^k\phi(x).$$

It's easiest to quantise with Heisenberg fields (so we don't pick up signs by differentiating), then pick an arbitrary time later on (J_i is conserved so it doesn't matter what time we pick). Substituting the Heisenberg fields in, we get:

$$-\frac{i}{2}\epsilon_{ijk}\int d^3\mathbf{x}\,\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{q}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} - a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\cdot$$
$$x^j(-iq^k)\left(a_{\mathbf{q}}e^{-iq\cdot x} - a^\dagger_{\mathbf{q}}e^{iq\cdot x}\right)$$

To get delta functions, absorb $(-ix^j)$ into the exponentials using derivatives:

$$-\frac{i}{2}\epsilon_{ijk}\int d^3\mathbf{x}\,\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\sqrt{\frac{E_\mathbf{p}}{E_\mathbf{q}}}\left(a_\mathbf{p}e^{-ip\cdot x}-a_\mathbf{p}^\dagger e^{ip\cdot x}\right)\cdot$$
$$q^k\left(a_\mathbf{q}\frac{\partial}{\partial q_j}e^{-iq\cdot x}+a_\mathbf{q}^\dagger\frac{\partial}{\partial q_j}e^{iq\cdot x}\right)$$

Now integrate over $\mathbf{x}$ to get delta functions:

$$=\frac{i}{2}\epsilon_{ijk}\int\frac{d^3\mathbf{p}}{(2\pi)^3}d^3\mathbf{q}\,\sqrt{\frac{E_\mathbf{p}}{E_\mathbf{q}}}q^k\Big((a_\mathbf{p}^\dagger a_\mathbf{q}^\dagger-a_\mathbf{p}a_\mathbf{q})\frac{\partial}{\partial q_j}(\delta^3(\mathbf{p}+\mathbf{q}))$$
$$+(a_\mathbf{p}^\dagger a_\mathbf{q}-a_\mathbf{p}a_\mathbf{q}^\dagger)\frac{\partial}{\partial q_j}(\delta^3(\mathbf{p}-\mathbf{q}))\Big).$$

Need to integrate by parts now. Easiest to define:

$$L_i^{(\mathbf{q})}=i\epsilon_{ijk}q^k\frac{\partial}{\partial q_j},$$

as in ordinary quantum mechanics. This operator clearly satisfies the Leibniz property, so can be used in integration by parts. Note also that $L_i^{(\mathbf{q})}f(\mathbf{q}^2)=0$, since acting with $\partial/\partial q_j$ produces something proportional to q^j, $L_i^{(\mathbf{q})}f(\mathbf{q}^2)\propto\epsilon_{ijk}q^jq^k=0$, so L operators pass straight through square root energy:

$$=\frac{1}{2}\int\frac{d^3\mathbf{q}}{(2\pi)^3}(a_{-\mathbf{q}}L_i^{(\mathbf{q})}a_\mathbf{q}-a_{-\mathbf{q}}^\dagger L_i^{(\mathbf{q})}a_\mathbf{q}^\dagger+a_\mathbf{q}L_i^{(\mathbf{q})}a_\mathbf{q}^\dagger-a_\mathbf{q}^\dagger L_i^{(\mathbf{q})}a_\mathbf{q}).$$

The first two terms are zero, because they are odd. This can be seen by sending $\mathbf{q}\mapsto-\mathbf{q}$ in these terms, then integrating by parts. The L_i are symmetric so are unchanged by the transformation.

Hence after normal ordering we have:

$$Q_i==\frac{1}{2}\int\frac{d^3\mathbf{q}}{(2\pi)^3}(L_i^{(\mathbf{q})}a_\mathbf{q}^\dagger a_\mathbf{q}-a_\mathbf{q}^\dagger L_i^{(\mathbf{q})}a_\mathbf{q}).$$

Integrate by parts on the first term, and we're done. □

This result allows us to determine the angular momentum of the quantum KG field.

Theorem: $Q_i\,|\mathbf{p}\rangle=L_i^\mathbf{p}\,|\mathbf{p}\rangle$. That is, KG particles are spinless.

Proof: We have:

$$Q_i\,|\mathbf{p}\rangle=-\int\frac{d^3\mathbf{q}}{(2\pi)^3}a_\mathbf{q}^\dagger L_i^{(\mathbf{q})}[\sqrt{2E_\mathbf{q}}a_\mathbf{q}a_\mathbf{p}^\dagger\,|0\rangle].$$

Integrate by parts to move L off the particle state. Then use commutation relations of a and $a^\dagger$ to create a δ function; the result follows. □

2.12 Causality

Definition: A theory is *causal* if spacelike separated operators commute. That is, if x and y are spacelike separated, then $[O_1(x),O_2(y)]=0$.

Theorem: In Klein-Gordon theory, $[\phi(x),\phi(y)]=0$ for spacelike separated x and y, and $[\phi(x),\phi(y)]\neq0$ for null or timelike separated x and y (i.e. the theory is causal).

Proof: Define $\Delta(x-y)=[\phi(x),\phi(y)]$. A short calculation shows:

$$\Delta(x-y)=\int\frac{d^3\mathbf{p}}{(2E_\mathbf{p})(2\pi)^3}\left(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\right).$$

This expression uses the Lorentz invariant integration measure, and only has 4-vector products in the integrand, so it's Lorentz invariant. For a timelike separation, we can transform to $x-y=(t,0,0,0)$ by Lorentz transformation, which gives a non-zero result when we insert into the integral.

For a spacelike separation, we can transform to a frame where the events x and y occur at equal times, so that $p\cdot(x-y)=-\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})$. Swapping $\mathbf{p}\mapsto-\mathbf{p}$ in the second term cancels the first (modulus of Jacobian of transformation is 1). □

2.13 Propagators

Make a particle at y. What's the probability we'll see it at x?

Definition: The *propagator* is defined by $D(x-y)=\langle0|\phi(x)\phi(y)|0\rangle$.

Theorem: An integral expression for the propagator is:

$$D(x-y)=\int\frac{d^3\mathbf{p}}{(2\pi)^32E_\mathbf{p}}e^{-ip\cdot(x-y)}.$$

Proof: Brief calculation. □

Note $D(x-y)$ does not vanish for $(x-y)^2<0$. However, $\Delta(x-y)=D(x-y)-D(y-x)=0$ does vanish as we saw above. Since there is no way to order spacelike separated events, it is just as probable for a particle to go from x to y as it is y to x.

For complex scalar fields, we see $[\psi(x),\psi^\dagger(y)]=0$ for x and y spacelike separated. By the same argument, this shows the amplitude for a particle to go from x to y cancels the amplitude for an *antiparticle* to go from y to x.

2.14 The Feynman propagator

Definition: The *Feynman propagator* is $\Delta_F(x-y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$, where T stands for *time-ordering*, given by:

$$T\{\phi(x)\phi(y)\} = \begin{cases} \phi(x)\phi(y) \text{ if } x^0 > y^0, \\ \phi(y)\phi(x) \text{ otherwise.} \end{cases}$$

Theorem: An integral expression for the Feynman propagator is

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2},$$

where integration along p^0 is defined by its analytic continuation in the complex plane, given by the contour:

Proof: Notice that $p^2-m^2 = (p^0)^2-E_{\mathbf{p}}^2 = (p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})$, so the poles are at $p^0 = \pm E_{\mathbf{p}}$ as expected. The residue at the poles is $\pm e^{\pm iE_{\mathbf{p}}(x^0-y^0)}/2E_{\mathbf{p}}$ respectively.

When $x^0 > y^0$, close in the LHP so we can use Jordan's Lemma. As $p^0 \to \infty$, we see we can apply Jordan's Lemma, so we get (in combination with the Residue Theorem, and a minus since the contour is anticlockwise):

$$\Delta_F(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^4}\frac{1}{2E_{\mathbf{p}}}(-2\pi i)ie^{-iE_{\mathbf{p}}(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}.$$

Hence $\Delta_F(x-y) = D(x-y)$. Similarly, for $x^0 < y^0$, get $\Delta_F(x-y) = D(y-x)$, so done. □

Using the contour above is equivalent to instead calculating the integral

$$\Delta_F(x-y) = \lim_{\epsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}.$$

Using this integral is called the $i\epsilon$ *prescription*. All it does is push the poles off the real axis slightly.

Theorem: The Feynman propagator is a Green's function for the Klein-Gordon operator.

Proof: We have:

$$\begin{aligned}(\partial_t^2 - \nabla^2 + m^2)\Delta_F(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{i(-p^2+m^2)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} \\ &= -i\int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)} = -i\delta^4(x-y). \quad \square\end{aligned}$$

3 Interacting scalar theory

3.1 Types of theory

Consider perturbations to the free Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=3}^{\infty}\frac{\lambda_n\phi^n}{n!}.$$

Begin with some dimensional analysis. Since $[S]=0$, we find $[\mathcal{L}]=4$. So $[\phi]=1$. This gives $[\lambda_n] = 4-n$. There are three cases:

- $4-n>0$, i.e. $n=3$. Then $[\lambda_3]=1$, so λ_3/E is dimensionless, where E is the energy scale. So for low energies these terms are important, but can be neglected at high energies. We call such terms are *relevant perturbations*, and the theories *renormalisable*.
- $4-n=0$, i.e. $n=4$. Then $[\lambda_4]=0$, so the perturbation is small for $\lambda_4 \ll 1$. These terms are called *marginal perturbations*. We call the theories *renormalisable*.
- $4-n<0$, i.e. $n>4$. Then $[\lambda_n]=4-n$, so $\lambda_4 E^{n-4}$ is dimensionless. This is small at low energies, and large at high energies. We call such terms *irrelevant perturbations*, and the theories *non-renormalisable*.

In this course we will only consider relevant and marginal perturbations.

3.2 The interaction picture

Write the Hamiltonian of an interacting theory as $H = H_0 + H_{\text{int}}$, where H_0 is the free theory Hamiltonian. In general:

Definition: *Interaction picture operators* are defined by $O_I(t) = e^{iH_0t}O_Se^{-iH_0t}$, where O_S is the Schrödinger operator. *Interaction picture states* are defined by $|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S$.

Theorem: $\dot{O}_I(t) = i[H_0, O_I(t)]$.

Proof: $\dot{O}_I(t) = iH_0e^{iH_0t}O_Se^{-iH_0t} - ie^{iH_0t}O_Se^{-iH_0t}H_0 = i[H_0, O_I(t)]$. □

Theorem: We have:

$$i\frac{d|\psi\rangle_I}{dt} = H_I(t)|\psi\rangle_I,$$

where $H_I = (H_{\text{int}})_I$, i.e. the interaction part of the Hamiltonian in the interaction picture.

Proof: We have:

$$\frac{d}{dt}(e^{iH_0t}|\psi\rangle_S) = ie^{iH_0t}H_0|\psi\rangle_S - ie^{iH_0t}H|\psi\rangle_S,$$

by the Schrödinger equation. Now note $-H_{\text{int}} = H_0 - H$, and so we put the pieces together to get the result. □

Slogan: Interaction operators evolve according to the **Heisenberg equation** with **free** Hamiltonian H_0. Interaction states evolve according to the **Schrödinger equation** with Hamiltonian H_I, i.e. just the **interaction part**.

Hence the field operators in the interaction picture are just the free field Heisenberg operators:

$$\phi_I(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right),$$

where $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ are the free creation and annihilation operators, still obeying $[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')$. Note $a_{\mathbf{p}}|0\rangle$ for the vacuum of the free theory, but $a_{\mathbf{p}}|\Omega\rangle \neq 0$, for the vacuum $|\Omega\rangle$ of the interacting theory.

3.3 Dyson's formula

We want to know how states evolve in interacting theory; to find out, define:

Definition: The *interaction time-evolution operator* is defined by $|\psi(t)\rangle_I = U(t,t_0)\,|\psi(t_0)\rangle_I$.

Another useful realisation of this operator is as the interaction picture version of the Schrödinger time evolution operator: $U(t,t_0) = e^{iH_0t'}U_S(t,t_0)e^{-iH_0t} = e^{iH_0t'}e^{-iH(t-t_0)}e^{-iH_0t}$.

Theorem: The time-evolution operator obeys the equation:

$$i\frac{dU(t,t_0)}{dt} = H_IU(t,t_0), \quad U(t,t) = I.$$

Proof: Boundary condition clear by Definition. For equation, note:

$$H_I\,|\psi(t)\rangle_I = i\frac{d}{dt}(U(t,t_0)\,|\psi(t_0)\rangle_I) = i\frac{dU(t,t_0)}{dt}\,|\psi(t_0)\rangle_I.$$

LHS equality follows from state evolution equation above. RHS equality is just differentiation. Write $|\psi(t)\rangle_I = U(t,t_0)\,|\psi(t_0)\rangle_I$, and compare left and right. $\square$

To solve, integrate directly iteratively: $U(t,t_0) =$

$$I + (-i)\int_{t_0}^{t} dt'\; H_I(t') + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\; H_I(t')H_I(t'') + \ldots$$

To get *Dyson's formula*, we write each of the multiple integrals in a clever way, using the diagram:

This diagram shows

$$\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\; H_I(t')H_I(t'') = \frac{1}{2}\int_{t_0}^{t} dt'\int_{t_0}^{t} dt''\; T\{H_I(t')H_I(t'')\},$$

where T stands for time-ordering. Similar diagrams show that the same thing happens for all other terms in the expansion. So we can always put a time-ordering inside the multiple integrals, then just integrate over the same range each time. Hence we have:

Dyson's formula:

$$U(t,t_0) = \mathrm{Texp}\left(-i\int_{t_0}^{t} dt'\; H_I(t')\right).$$

3.4 Scattering and meson decay example

In scattering problems, we assume that the initial and final states are well-separated, so are essentially non-interacting. Thus we assume the initial and final states are *eigenstates of the free theory*.

Definition: The S*-matrix* is defined as:

$$S = \lim_{\substack{t\to\infty \\ t_0\to-\infty}} U(t,t_0).$$

The amplitude for an initial state $|i\rangle$ to scatter into a final state $|f\rangle$ is given by $\langle f|S|i\rangle$.

Example: Consider *scalar Yukawa theory*:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \partial_\mu\psi\partial^\mu\psi^* - \mu^2\psi\psi^* - g\psi^*\psi\phi.$$

Consider meson decay into a nucleon and an anti-nucleon: $|i\rangle = \sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^\dagger|0\rangle$, $|f\rangle = \sqrt{4E_{\mathbf{q}_1}E_{\mathbf{q}_2}}b_{\mathbf{q}_1}^\dagger c_{\mathbf{q}_2}^\dagger|0\rangle$ (note these are in the free theory).

By Dyson's formula, we have to first order in g:

$$\begin{aligned}\langle f|S|i\rangle &= \langle f|\mathrm{Texp}\left(-i\int d^4x\; g\psi^*(x)\psi(x)\phi(x)\right)|i\rangle \\ &= -ig\,\langle f|\int d^4x\; T\{\psi^*(x)\psi(x)\phi(x)\}|i\rangle + O(g^2) \\ &= -ig\,\langle f|\int d^4x\; \psi^*(x)\psi(x)\phi(x)|i\rangle + O(g^2),\end{aligned}$$

since there's only one time in the first order integral, so the time-ordering is trivial. Note also that the zeroth-order term vanishes.

We'll see how to calculate such integrals in a nice way shortly.

3.5 Wick's theorem

From Dyson's formula, we know we want a quick way of computing quantities like $\langle f|T\{H_I(x_1)...H_I(x_n)\}|i\rangle$.

Definition: A *contraction* of a pair of fields in a string $\phi(x_1)\phi(x_2)...\phi(x_n)$ is defined by:

$$\phi(x_1)...\overbracket{\phi(x_i)...\phi(x_j)}...\phi(x_n) :=$$
$$\Delta_F(x_i - x_j)\phi(x_1)...\phi(x_{i-1})\phi(x_{i+1})...\phi(x_{j-1})\phi(x_{j+1})...\phi(x_n).$$

Wick's Theorem: The time-ordering of a string of fields may be written as:

$$T\{\phi(x_1)...\phi(x_n)\} =: \phi(x_1)...\phi(x_n) : + : \text{all contractions} :$$

Proof (Sketch): By induction. The base case is for two fields, i.e. $T\{\phi(x)\phi(y)\}$. Write:

$$\phi^+(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}a_\mathbf{p}e^{-ip\cdot x},$$
$$\phi^-(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}a^\dagger_\mathbf{p}e^{ip\cdot x}.$$

Assume $x^0 > y^0$. Then

$$T\{\phi(x)\phi(y)\} = (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y))$$

$$= \overbrace{\phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y)}^{\text{normal ordered string}}$$
$$+ \underbrace{[\phi^+(x), \phi^-(y)]}_{\text{the propagator}}.$$

Recall $D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi^+(x)\phi^-(y)|0\rangle = \langle 0|[\phi^+(x),\phi^-(y)]|0\rangle = [\phi^+(x),\phi^-(y)]$ (since ϕ^- is a creation operator and ϕ^+ is an annihilation operator). Thus

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + D(x-y).$$

Swapping x and y in the above gives $T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + D(y-x)$, since normal ordering is symmetric under the interchange from above expression. Thus

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + \underbrace{\Delta_F(x-y)}_{\text{contraction}}.$$

We sketch the induction part. Write $\phi(x_i) = \phi_i$ for simplicity. Now suppose Wick's Theorem holds for $T\{\phi_2...\phi_n\}$. Consider $T\{\phi_1\phi_2...\phi_n\}$, and suppose $x_1^0 > x_k^0$. Then:

$$T\{\phi_1\phi_2...\phi_n\} = \phi_1 T\{\phi_2...\phi_n\}$$
$$= (\phi_1^+ + \phi_1^-)(: \phi_2...\phi_n : + : \text{all contractions} :).$$

The ϕ_1^- can stay where it is, but the ϕ_1^+ has to commute past the normal-ordered string all the way to the RHS. Every time it commutes past an operator in the normal ordered string, we get a propagator $D(x_1 - x_k)$. After time ordering this gives $\Delta_F(x_1 - x_k)$. □

Wick's Theorem also holds for complex scalar fields. Here, we define contractions via:

$$\overbracket{\psi(x)\psi^*(y)} = \Delta_F(x-y), \quad \overbracket{\psi(x)\psi(y)} = 0 = \overbracket{\psi^*(x)\psi^*(y)}.$$

Here, the Feynman propagator is the nucleon propagator (i.e. replace m with μ).

3.6 Example: nucleon-nucleon scattering

Work in scalar Yukawa theory. Consider nucleon to nucleon scattering: $|i\rangle = \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}b^\dagger_{\mathbf{p}_1}b^\dagger_{\mathbf{p}_2}|0\rangle = |p_1, p_2\rangle$, $|f\rangle = \sqrt{4E_{\mathbf{p}'_1}E_{\mathbf{p}'_2}}b_{\mathbf{p}'_1}b_{\mathbf{p}'_2}|0\rangle = |p'_1, p'_2\rangle$. We are interested only in the case when the particles interact, i.e. we should consider $\langle f|(S-I)|i\rangle$.

By Dyson's formula, we get:

$$\langle f|(S-I)|i\rangle = -ig\,\langle f|\int d^4x\ \psi^*(x)\psi(x)\phi(x)|i\rangle +$$
$$\frac{(-ig)^2}{2}\langle f|\int d^4x_1d^4x_2\ T\{\psi^*(x_1)\psi(x_1)\phi(x_1)\psi^*(x_2)\psi(x_2)\phi(x_2)\}|i\rangle\,,$$

up to order $O(g^2)$. Note the order $O(g)$ term cancels, since $\phi(x) \sim a + a^\dagger$, and a annihilates the $|i\rangle$ state, and $a^\dagger$ annihilates the $\langle f|$ state. So lowest order non-zero contribution is order $O(g^2)$.

Use Wick's Theorem to evaluate the time-ordered middle part. Note that if the ϕ's are not contracted, they are in the normal-ordered part, and we get complete annihilation as at order $O(g)$. So must contract ϕ's.

We note that we need exactly two b's on the right, for if we had more, we could commute two past the $b^\dagger$'s in $|i\rangle$, and annihilate the $|0\rangle$. If we had less, we'd have enough $b^\dagger$'s on the right to commute to the left, and with the left overs annihilating $\langle 0|$. Similarly, we need exactly two $b^\dagger$'s on the left. So we mustn't contract the ψ's and ψ^*'s.

Hence we only need the terms:

$$: \psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2) : \overbracket{\phi(x_1)\phi(x_2)},$$

and the same term with $x_1 \leftrightarrow x_2$.

Recalling how many b's and $b^\dagger$'s we need, we see that: $\langle f| : \psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2) : |i\rangle$

$$= \int \frac{d^3\mathbf{q}_1...d^3\mathbf{q}_4\ \sqrt{16E_{\mathbf{p}_1}...E_{\mathbf{p}_2}}}{(2\pi)^{12}\sqrt{2E_{\mathbf{q}_1}...2E_{\mathbf{q}_4}}}\langle 0|b_{\mathbf{p}'_1}b_{\mathbf{p}'_2}b^\dagger_{\mathbf{q}_1}b^\dagger_{\mathbf{q}_2}b_{\mathbf{q}_3}b_{\mathbf{q}_4}b^\dagger_{\mathbf{p}_1}b^\dagger_{\mathbf{p}_2}|0\rangle$$
$$\cdot e^{i(q_1\cdot x_1 + q_2\cdot x_2 - q_3\cdot x_1 - q_4\cdot x_2)}.$$

Using the commutation relations, we can evaluate the inner product in the integral: $\langle 0|b_{\mathbf{p}'_1}b_{\mathbf{p}'_2}b^\dagger_{\mathbf{q}_1}b^\dagger_{\mathbf{q}_2}b_{\mathbf{q}_3}b_{\mathbf{q}_4}b^\dagger_{\mathbf{p}_1}b^\dagger_{\mathbf{p}_2}|0\rangle$

$$= (2\pi)^{12}(\delta^3(\mathbf{p}'_1 - \mathbf{q}_2)\delta^3(\mathbf{p}'_2 - \mathbf{q}_1) + \delta^3(\mathbf{p}'_1 - \mathbf{q}_1)\delta^3(\mathbf{p}'_2 - \mathbf{q}_2))$$
$$\cdot(\delta^3(\mathbf{q}_4 - \mathbf{p}_1)\delta^3(\mathbf{q}_3 - \mathbf{p}_2) + \delta^3(\mathbf{q}_4 - \mathbf{p}_2)\delta^3(\mathbf{q}_3 - \mathbf{p}_1)).$$

Putting all this back into the integral, we have: $\langle f| : \psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2) : |i\rangle$

$$= (e^{i(p_1' \cdot x_2 + p_2' \cdot x_1)} + e^{i(p_2' \cdot x_2 + p_1' \cdot x_1)})$$
$$\cdot (e^{-i(p_1 \cdot x_2 + p_2 \cdot x_1)} + e^{-i(p_1 \cdot x_1 + p_2 \cdot x_2)}).$$

Hence our expression for $\langle f|(S-I)|i\rangle$ is

$$\frac{(-ig)^2}{2} \int d^4x_1 d^4x_2 \Big(e^{ix_2 \cdot (p_1' - p_1) + ix_1 \cdot (p_2' - p_2)}$$
$$+ e^{ix_2 \cdot (p_2' - p_1) + ix_1 \cdot (p_1' - p_2)} + (x_1 \leftrightarrow x_2) \Big) \int \frac{d^4k}{(2\pi)^4} \frac{ie^{ik \cdot (x_2 - x_1)}}{k^2 - m^2 + i\epsilon}.$$

Since integral is symmetric on x_1 and x_2, can swap in second term, so just get twice first term. Thus left with final answer:

$$i(-ig)^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \Big(\frac{i}{(p_1 - p_1')^2 - m^2 + i\epsilon}$$
$$+ \frac{i}{(p_2' - p_1)^2 - m^2 + i\epsilon} \Big).$$

3.7 Feynman diagrams and rules

Whilst the calculation using Wick's Theorem is simpler, it is still horrible. We therefore use a *diagrammatic* method for calculating scattering amplitudes: *Feynman diagrams*.

Feynman diagrams:

1. Draw an external line for each particle in $|i\rangle$ and in $|f\rangle$. Draw dashed lines for the real scalar fields ϕ, and solid lines for the complex scalar fields, ψ. Add arrows for complex fields to show the flow of charge. Draw an in-going arrow for an initial particle and an out-going arrow for an initial anti-particle. Do the opposite for final particles/anti-particles.
2. Join the lines together at vertices. We can only join vertices in a way the the interaction term in the Lagrangian allows. For example, if the interaction was $\phi^5\psi^*\psi$, we would be only be able to join 5 real scalar lines, and two complex scalar lines, at a vertex. No other vertices are permitted.
3. We can, however, have as many vertices as we like. For example, in scalar Yukawa theory (i.e interaction $\psi^*\psi\phi$) both of the following are acceptable diagrams:

We associate Feynman diagrams to terms in the amplitude, $\langle f|(S-I)|i\rangle$. Our interpretation of a Feynman diagram is as follows:

- Each vertex in a Feynman diagram represents an integration variable x_1. For example, a two-vertex diagram corresponds to the second term in the amplitude, meaning an integral over $\int d^4x_1 d^4x_2$.
- Connecting vertices with edges corresponds to contraction in the Wick expansion (note the external particle lines don't have vertices on the outside). The remaining fields are included in the normal-ordered part.
- Recall that the fields in the normal-ordered part need to take a very specific form; namely, they need to annihilate the incoming particles, and produce the outgoing particles.

  After doing all our commutation work, this amounts to putting in a factor of $e^{-ip\cdot x_1}$ for a particle with momentum p coming into a vertex x_1, and $e^{ip\cdot x_1}$ for a particle with momentum p going out of a vertex x_1.

For example, the diagram:

Corresponds to the term:

$$(-ig)^2 \int d^4x_1 d^4x_2 \, e^{ix_2\cdot(p_1' - p_1) + ix_1 \cdot (p_2' - p_2)} \Delta_F(x_1 - x_2).$$

(*Note:* There is no factor of 2 here, because by convention, the vertices of a Feynman diagram are unlabelled. This means that a Feynman diagram represents *both* the diagram with the vertices in one position, and with the vertices interchanged, if this is possible.)

Note that given an amplitude of this form, we can integrate out x_1, x_2, etc. In particular, this gives delta functions which fix the values of the momentum in the propagators (if the propagators don't form a loop). That is, momentum conservation is imposed at each vertex by the integration over the vertices.

Hence, we can associate a *number* to each Feynman diagram via the *Feynman rules*:

The Feynman rules:

1. At every vertex, write down a factor $(-ig)$, where g was the coupling of the interaction.
2. Add a factor of $(2\pi)^4\delta^4\left(\sum_{\text{ingoing}} p_i - \sum_{\text{outgoing}} p_i\right)$ for global momentum conservation.
3. Impose 4-momentum conservation at each vertex in the diagram. Write down a propagator:
$$\frac{i}{k^2 - m^2 + i\epsilon}$$
for each internal line (replace m by μ for internal ψ fields), inserting the determined k, or leaving until Step 4.
4. Integrate over any undetermined momenta k.

Example: For nucleon-nucleon scattering in scalar Yukawa theory, there are two possible diagrams:

Using the diagrams and the Feynman rules, we can just write down the answer we got above.

In practice, the Feyman rules are refined to instead find the *scattering amplitude* of the process:

Definition: Write

$$\langle f|(S-I)|i\rangle = iA_{fi}(2\pi)^4\delta^4\left(\sum_{\text{initial}} p_i - \sum_{\text{final}} p_f\right).$$

The *scattering amplitude* is A_{fi}.

The Feynman rules are easily refined to compute iA_{fi}:

1. Write a factor of $(-ig)$ at each vertex.
2. Impose 4-momentum conservation at each vertex.
3. For each internal line, write a factor of the propagator.
4. Integrate over any undetermined momenta.

3.8 Many examples

Example 1: For $\psi(p_1)\overline{\psi}(p_2) \to \phi(p_1')\phi(p_2')$ scattering in scalar Yukawa theory, the lowest order amplitude is:

$$iA_{fi} = (-ig)^2\left(\frac{i}{(p_1-p_1')^2-\mu^2} + \frac{i}{(p_1-p_2')^2-\mu^2}\right).$$

Example 2: Consider $\phi(p_1)\phi(p_2) \to \phi(p_1')\phi(p_2')$ scattering in ϕ^4 theory, i.e. with interaction $\mathcal{L}_{\text{int}} = -\lambda\phi^4/4!$. The lowest order amplitude is just $-i\lambda$.

We lose the $4!$ because the diagram represents the term:
$$\frac{-i\lambda}{4!}\int d^4x\ \langle f| : \phi(x)\phi(x)\phi(x)\phi(x) : |i\rangle$$
in the Wick expansion. When we write out this in terms of annihilation and creation operators, we need, as before, exactly two annihilation operators to act on $|i\rangle$ and two creation operators to act to the left on $|f\rangle$.

How many ways of making this choice are there? Since any of the field operators can contribute any of the annihilation/creation operators, and order matters, there are $4!$ ways, which gives $4!$ terms in the normal-ordered expansion, which cancels the $4!$ up front.

Example 3: Consider a theory of 3 fields governed by the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{3}\left(\frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2\phi_i^2\right) - \frac{1}{8}\lambda\left(\sum_{i=1}^{3}\phi_i^2\right)^2,$$

with $[\phi_i,\phi_j]=0$. Then the propagator can be calculated. If $x^0 > y^0$, we have for $i \neq j$:

$$\langle 0|T\{\phi_i(x)\phi_j(y)\}|0\rangle = \langle 0|\phi_i(x)\phi_j(y)|0\rangle = \langle 0|[\phi_i(x),\phi_j(y)]|0\rangle = 0,$$

since all annihilation/creation operators commute if $i \neq j$, so can readily annihilate $|0\rangle$ and $\langle 0|$. For i equal to j, just get normal propagator. Hence: $\langle 0|T\{\phi_i(x)\phi_j(y)\}|0\rangle = \delta_{ij}\Delta_F(x-y)$.

The interaction term gives two possible interactions: $-\lambda\phi_i^4/8$ interactions and $-\lambda\phi_i^2\phi_j^2/4$ interactions (for $i \neq j$).

To lowest order, $\phi_i\phi_i \to \phi_i\phi_i$ scattering has amplitude $-3i\lambda$, using the reasoning from ϕ^4 theory.

To lowest order, $\phi_i\phi_j \to \phi_i\phi_j$ scattering has amplitude $-i\lambda$. This is because in $:\phi_i(x)\phi_i(x)\phi_j(x)\phi_j(x):$ we are now more restricted as to where the fields can go; an i and j must act to the left, and an i and j must act to the right. So pick from 2 possible i, and 2 possible j, giving a factor $2\times 2 = 4$.

Example 4: Consider a theory governed by the Lagrangian:

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - \mu^2 \psi^* \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$
$$-g\psi^*\psi\phi - h|\psi|^4 - k\phi^3 - l\partial_\mu \psi \partial^\mu \psi^* \phi.$$

Let's calculate the vertex rules for this theory.

- For $\psi^*\psi\phi$, the amplitude is $-ig$, because there is no symmetry of the operators at a vertex.
- For ϕ^3 vertices, we have amplitude $-6ik$, because of the obvious factor $3!$.
- For $|\psi|^4 = (\psi^*)^2\psi^2$ vertices, we have amplitude $-4ih$, as with $\phi_i\phi_j \to \phi_i\phi_j$ scattering above.
- For $\partial_\mu\psi\partial^\mu\psi^*\phi$, consider a vertex such as $\phi(p_1) \to \psi(p_2)\psi^*(p_3)$. From our interpretation of the Feynman diagram, we expect this to give an $e^{-ip_1 \cdot x}$ factor in position space from the ϕ, but due to the derivatives, we expect a $-p_2 \cdot p_3 e^{i(p_2+p_3)\cdot x}$ from the $\psi^*\psi$. Hence the amplitude is $ilp_2 \cdot p_3$. Note that this can change sign dependent on whether the nucleons are both ingoing/outgoing or one ingoing, other outgoing.

Note that $-ig$ and $ilp_2 \cdot p_3$ combine additively at a $\psi^*\psi\phi$ vertex.

3.9 Correlation functions

We won't really use correlation functions in this course, but they will be important in the future. Throughout this section, work in ϕ^4 theory.

Definition: Functions of the form

$$\langle 0|T\{\phi(x_1)...\phi(x_m)S\}|0\rangle\,,$$

are called *correlation functions*. Using the expansion for S, a general term in this may be written (using $\phi(x_i) = \phi_i$):

$$\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int d^4y_1...d^4y_n\, \langle 0|T\{\phi_1...\phi_m\phi^4(y_1)...\phi^4(y_n)\}|0\rangle\,.$$

Since this is sandwiched between vacuum states, any normal ordered part from Wick's Theorem vanishes. So we must perform all possible contractions in all possible ways.

Example: Consider $n = 1$, $m = 4$. We can contract in the following ways:

- Contract each ϕ_i with a $\phi(x)$. ϕ_1 can be paired with any of the 4 $\phi(x)$'s, ϕ_2 can be paired with any of the remaining 3 $\phi(x)$'s, etc, so there are $4!$ terms of this type. So the contribution is:

  $$-i\lambda \int d^4x\, \Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x_3-x)\Delta_F(x_4-x).$$

- Contract two ϕ_i fields, and two $\phi(x)$'s, then contract remaining ϕ_i's with $\phi(x)$'s. There are $\binom{4}{2}$ ways of picking which ϕ_i's we'll contract, and $\binom{4}{2}$ ways of picking which $\phi(x)$'s we'll contract. There are then 2 ways of pairing up the remaining fields. So there are a total of $6 \times 6 \times 2 = 72$ terms of this type. They look like:

  $$-\frac{i\lambda}{2}\Delta_F(x_1-x_2)\int d^4x\, \Delta_F(x_3-x)\Delta_F(x_4-x)\Delta_F(x-x),$$

  and 5 other similar terms with x_1, x_2, x_3, x_4 permuted. Note $\Delta_F(x-x) = \Delta_F(0) = \infty$, which we will ignore.

- Contract all ϕ_i's and all $\phi(x)$'s separately. There are $3 \times 3 = 9$ ways of doing this (since we pick a partner for ϕ_1 from 3 other fields, then the other pairing is determined; similarly for $\phi(x)$'s). Terms look like:

  $$-\frac{i\lambda}{8}\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)\int d^4x\, \Delta_F(x-x)\Delta_F(x-x),$$

  and two other similar terms with other pairings of ϕ_i's.

This calculation can be represented by the diagrammatic expansion:

Definition: The constant we divide by is called the *symmetry factor* of a diagram. The symmetry factors are 1, 2 and 8 above.

Example: Consider $n = 2$, $m = 4$, and consider the term with contractions:

$$\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y).$$

We can represent this as a diagram:

We now compute the diagram's symmetry factor. From the Wick expansion, we get a constant $\frac{1}{2}\left(\frac{1}{4!}\right)^2$. We lose the $1/2$ immediately because of exchange of integration variables: $x \leftrightarrow y$.

The other symmetries of this expression are:

- ϕ_1 connects to a $\phi(x)$ - 4 choices.
- ϕ_2 connects to a $\phi(x)$ - 3 remaining choices.
- ϕ_3 connects to a $\phi(y)$ - 4 choices.
- ϕ_4 connects to a $\phi(y)$ - 3 remaining choices.
- $\phi(x)$ connects to a $\phi(y)$ - 2 choices.

Final $\phi(x)$, $\phi(y)$ pair is then determined. So $4 \times 3 \times 4 \times 3 \times 2 = (4!)^2/2$. Hence the symmetry factor of the diagram is 2.

We can make calculations easier by introducing *position space Feynman rules* for correlation functions:

1. Write a factor of $-i\lambda \int d^4x$ for each vertex x.
2. Write a factor of $\Delta_F(y-z)$ for each line from y to z.
3. Divide by the symmetry factor of the diagram.

Note we don't need to worry about $e^{\pm ip\cdot x}$'s because there are no external edges; we work with vacuum to vacuum calculations.

We can easily upgrade these to momentum space rules:

1. Write a factor of $-i\lambda$ for each vertex.
2. Impose four-momentum conservation at each vertex.
3. Write a factor of the propagator $\dfrac{i}{p^2 - m^2 + i\epsilon}$ for each internal edge with momentum p.
4. Integrate over any undetermined momenta.
5. Divide by the symmetry factor of the diagram.

3.10 Computing symmetry factors

There is a recipe for computing symmetry factors from the *diagrams* rather than the Wick expansion.

1. If a propagator starts and ends at the same vertex, get a factor of 2.
2. If a pair of vertices is connected by k identical propagators, get a factor of $k!$.
3. If vertices can be permuted without affecting the diagram, get a factor of the number of permutations.
4. If there are n identical disconnected pieces, get a factor of $n!$.
5. In a one-vertex subdiagram of the form:

   get an additional factor of 2.

Example: Consider the basketball diagram

This has symmetry factor $4! \times 2 = 48$ from rules 2 and 3.

3.11 Vacuum bubbles

Consider $\langle 0|S|0\rangle$. Its diagrammatic expansion is:

Note all of these diagrams have no external lines.

Definition: Diagrams with no external lines are called *vacuum bubbles*.

Theorem: We can write $\langle 0|S|0\rangle$ as:

$$\langle 0|S|0\rangle = \exp\left(\sum \text{distinct connected vacuum bubble types}\right).$$

Proof: Too hard for this course. We can verify it to second order, however, by checking the following:

Next term, we will see that

$$\langle \Omega|T\{\phi_1 ... \phi_m\}S|\Omega\rangle = \left(\sum \text{connected diagrams}\right) \cdot \langle 0|S|0\rangle,$$

where *connected* means every part of the diagram is connected to external points. For example, this diagram is still connected:

3.12 Green's functions

So far, we've neglected the *true vacuum* of interacting theory: $|\Omega\rangle$, obeying $H|\Omega\rangle = 0$ and $\langle\Omega|\Omega\rangle = 1$ (we worked with $|0\rangle$ obeying $H_0|0\rangle = 0$ and $\langle 0|0\rangle = 1$).

Definition: The *Green's function* is define by

$$G^{(n)}(x_1, ..., x_n) = \langle\Omega|T\{\phi_H(x_1)...\phi_H(x_n)\}|\Omega\rangle\,,$$

where ϕ_H are Heisenberg fields.

Theorem: We have:

$$\langle\Omega|T\{\phi_H(x_1)...\phi_H(x_m)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I(x_1)...\phi_I(x_m)S\}|0\rangle}{\langle 0|S|0\rangle}$$
$$= \sum (\text{connected diagrams with } m \text{ external points})\,.$$

Interpretation: If we discard Feynman diagrams with vacuum bubbles, we get the right answers (i.e. with the true vacuum) by working with the free vacuum.

Proof: Work from the RHS to the LHS. First assume WLOG that $x_1^0 > x_2^0 > ... > x_m^0$ (and write $x_i^0 = t_i$) so we can forget about time-ordering. Then the RHS numerator is:

$$\langle 0|U(\infty, t_1)\phi_I(x_1)U(t_1, t_2)...U(t_{m-1}, t_m)\phi_I(x_m)U(t_m, -\infty)|0\rangle$$

where we've just converted the S-matrix into time-ordered form. Converting the numerator of RHS to Heisenberg picture:

$$\underbrace{\langle 0|U(\infty, 0)\phi_H(x_1)\phi_H(x_2)...\phi_H(x_m)}_{\langle\psi|}U(0, -\infty)|0\rangle$$

Define the state $|\psi\rangle$ as shown above. We want to take the limit as $t_0 \to \infty$ in:

$$\langle\psi|U(0, t_0)|0\rangle = \langle\psi|U_S(0, t_0)|0\rangle\,,$$

where U_S is the Schrödinger time evolution operator. This holds since H_0 annihilates $|0\rangle$.

Insert an identity operator I using resolution of identity. Note that $U_S(0, t_0)|\Omega\rangle = |\Omega\rangle$, since the full Hamiltonian annihilates the interacting vacuum, and U_S is an exponential of the full Hamiltonian. Hence:

$$\langle\psi|U_S(0, t_0)|0\rangle =$$
$$\langle\psi|\,U_S(0, t_0)\underbrace{\left[|\Omega\rangle\langle\Omega| + \sum_{n=1}^{\infty}\int\prod_{j=1}^{n}\frac{d^3\mathbf{p}_j}{2E_{\mathbf{p}_j}(2\pi)^3}|p_1...p_n\rangle\langle p_1...p_n|\right]}_{=I}|0\rangle$$
$$= \langle\psi|\Omega\rangle\langle\Omega|0\rangle + \sum_{n=1}^{\infty}\int\prod_{j=1}^{n}\frac{d^3\mathbf{p}_j}{(2E_{\mathbf{p}_j}(2\pi)^3}\exp\left(i\sum_{k=1}^{n}E_{p_k}t_0\right),$$

since $|p_1...p_n\rangle$ are interaction eigenstates, and $U_S(0, t_0) = e^{iHt_0}$. As $t_0 \to -\infty$, the second term vanishes by the Riemann-Lebesgue Lemma:

$$\lim_{\mu\to\infty}\int_a^b f(x)e^{i\mu x} = 0,$$

for f absolutely integrable.

Hence $\langle\psi|U(0, t_0)|0\rangle = \langle\psi|\Omega\rangle\langle\Omega|0\rangle$. So our numerator reduces to:

$$\langle 0|U(\infty, 0)\phi_H(x_1)...\phi_H(x_m)|\Omega\rangle\langle\Omega|0\rangle\,.$$

Similarly, $\langle 0|U(\infty, 0)|\psi'\rangle = \langle 0|\Omega\rangle\langle\Omega|\psi'\rangle$, so the numerator reduces completely to:

$$\langle\Omega|\phi_H(x_1)...\phi_H(x_m)|\Omega\rangle\langle\Omega|0\rangle\langle 0|\Omega\rangle\,.$$

Finally, note the same argument applies to the denominator too: $\langle 0|S|0\rangle = \langle 0|U(\infty, 0)U(0, -\infty)|0\rangle = \langle 0|\Omega\rangle\langle\Omega|0\rangle$. Hence we get the LHS. □

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4 Cross-sections and decay rates

4.1 The Mandelstam variables

Definition: Defined the *Mandelstam variables* in a 2 to 2 scattering process by:

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_1')^2, \quad u = (p_1 - p_2')^2.$$

Theorem: $s + t + u$ is equal to the sum of the squares of the initial and final masses.

Proof: Sum is $3m_1^2 + m_2^2 + m_1'^2 + m_2'^2 + 2p_1 \cdot (p_2 - p_1' - p_2')$, which gives result on using momentum conservation. □

In particular, the Mandelstam variables are *not* independent of one another.

4.2 Cross sections

In real life, we don't have momentum eigenstates as our initial states; instead we have sharply peaked superpositions:

$$|i\rangle = \int\frac{d^3\tilde{\mathbf{p}}_1}{(2\pi)^3 2E_1}\frac{d^3\tilde{\mathbf{p}}_2}{(2\pi)^3 2E_2}f_1(\tilde{p}_1)f_2(\tilde{p}_2)|\tilde{p}_1\tilde{p}_2\rangle\,.$$

We assume the distributions f_i are sharply peaked at some $\tilde{p}_i = p_i$.

The *outgoing* particles $|f\rangle$ are still considered momentum eigenstates; this is a good approximation for collider experiments.

The transition probability for 2 to n scattering, with initial momenta p_1, p_2 (sharply peaked around these momenta) and final momenta q_i, is given by:

$$W = |\langle f|(S - I)|i\rangle|^2.$$

Inserting the integral form of $|i\rangle$, we have $W =$

$$(2\pi)^8 \int \frac{d^3\tilde{\mathbf{p}}_1}{(2\pi)^3 2E_1} \frac{d^3\tilde{\mathbf{p}}_2}{(2\pi)^3 2E_2} \frac{d^3\tilde{\mathbf{p}}'_1}{(2\pi)^3 2E'_1} \frac{d^3\tilde{\mathbf{p}}'_2}{(2\pi)^3 2E'_2} |A_{fi}|^2 f_1(\tilde{p}_1)$$
$$f_1^*(\tilde{p}'_1) f_2(\tilde{p}_2) f_2^*(\tilde{p}'_2) \delta^4\left(\sum_i q_i - \tilde{p}_1 - \tilde{p}_2\right) \delta^4\left(\sum_i q_i - \tilde{p}'_1 - \tilde{p}'_2\right)$$

Expand the second delta function explicitly, and use $\sum_i q_i = p_1 + p_2 \approx \tilde{p}_1 + \tilde{p}_2$ to write $W =$

$$\int d^4x \int \frac{d^3\tilde{\mathbf{p}}_1}{(2\pi)^3 2E_1} f_1(\tilde{p}_1) e^{i\tilde{p}_1\cdot x} \int \frac{d^3\tilde{\mathbf{p}}'_1}{(2\pi)^3 2E'_1} f_1^*(\tilde{p}'_1) e^{-i\tilde{p}'_1\cdot x}$$
$$\int \frac{d^3\tilde{\mathbf{p}}_2}{(2\pi)^3 2E_2} f_2(\tilde{p}_2) e^{i\tilde{p}_2\cdot x} \int \frac{d^3\tilde{\mathbf{p}}'_2}{(2\pi)^3 E'_2} f_2^*(p'_2) e^{-i\tilde{p}'_2\cdot x}$$
$$\left((2\pi)^4 \delta^4\left(\sum_i q_i - \tilde{p}_1 - \tilde{p}_2\right) |A_{fi}|^2\right).$$

Recall the formula for the wavefunction:

$$\psi_i(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} e^{ip\cdot x} f_i(p).$$

Inserting into the above:

$$W = \int d^4x \frac{1}{\sqrt{2E_1}} \frac{1}{\sqrt{2E'_1}} \frac{1}{\sqrt{2E_2}} \frac{1}{\sqrt{2E'_2}} |\psi_1(x)|^2 |\psi_2(x)|^2$$
$$(2\pi)^4 |A_{fi}|^2 \delta^4\left(\sum_i q_i - \tilde{p}_1 - \tilde{p}_2\right).$$

Using $\tilde{p}_1 + \tilde{p}_2 \approx p_1 + p_2$, and $E_1 \approx E'_1$, $E_2 \approx E'_2$, we find that

$$\frac{dW}{d^4x} = \frac{|\psi_1(x)|^2}{2E_1} \frac{|\psi_2(x)|^2}{2E_2} (2\pi)^4 \delta^4\left(\sum_i q_i - p_1 - p_2\right) |A_{fi}|^2.$$

This is the *transition probability per unit time*.

We now convert this into something measurable. Suppose that we are in particle 1's rest frame, and that it has an effective cross-sectional area $d\sigma$. Let $\rho = |\psi_1(x)|^2$ be the probability density of the target particle, and ϕ be the flux, i.e. the probability density passing the point per unit time. Here, $\phi = |\psi_2(x)|^2 v$, where v is the relative velocity of the particles. Hence we have: $dW/d^4x = d\sigma\rho\phi$, and so

$$d\sigma = \frac{(2\pi)^4}{\mathcal{F}} \delta^4\left(p_1 + p_2 - \sum_i^n q_i\right) |A_{fi}|^2,$$

where $\mathcal{F}$ is the *flux factor*, $\mathcal{F} = 4E_1E_2v$.

Theorem: $\mathcal{F} = 4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}$.

Proof: Work in rest frame of second particle, i.e. $p_2 = (m_2, 0)$, $p_1 = (\sqrt{m_1^2 + p_1^2}, \mathbf{p}_1)$. The relative velocity is then $v = |\mathbf{p}_1|/E_1$, and we get the result using $E_1^2 = m_1^2 + |\mathbf{p}_1|^2$. The answer is Lorentz invariant so holds in all frames. □

Hence we have a final result:

Total cross-section: The total cross-section is $\sigma =$

$$\int \prod_{i=1}^n \left(\frac{d^3\mathbf{q}_i}{(2\pi)^3 2E_{\mathbf{q}_i}}\right) \frac{|A_{fi}|^2}{\mathcal{F}} (2\pi)^4 \delta^4\left(p_1 + p_2 - \sum_i^n q_i\right),$$

where $\mathcal{F} = 4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}$ is the *flux factor*.

4.3 2 to 2 scattering

In 2 to 2 scattering, we use the Mandelstam variables. Note:

$$t = m_1^2 + m_1'^2 - 2E_{\mathbf{p}_1}E_{\mathbf{q}_1} + 2\mathbf{p}_1\cdot\mathbf{q}_1 \Rightarrow \frac{dt}{d\cos(\theta)} = 2|\mathbf{p}_1||\mathbf{q}_1|,$$

where θ is the (frame-dependent) angle between $\mathbf{p}_1$ and $\mathbf{q}_1$, i.e. it is the *scattering angle*.

To simplify our general calculation, write

$$\frac{d^3\mathbf{q}_2}{2E_{\mathbf{q}_2}} = d^4q_2\ \delta(q_2^2 - m_2'^2) H(q_2^0),$$

where H is the Heaviside function (this is the reverse of the calculation showing Lorentz invariance of the measure). Expand the $\mathbf{q}_1$ integral in polars, then in terms of ϕ, t and energy $E_{\mathbf{q}_1}$:

$$\frac{d^3\mathbf{q}_1}{2E_{\mathbf{q}_1}} = \frac{|\mathbf{q}_1|^2 d|\mathbf{q}_1| d\cos(\theta) d\phi}{2E_{\mathbf{q}_1}} = \frac{1}{4|\mathbf{p}_1|} dE_{\mathbf{q}_1} d\phi dt.$$

Using the total cross-section formula above, and performing the q_2 and ϕ integrals, we find

$$\frac{d\sigma}{dt} = \frac{1}{8\pi\mathcal{F}|\mathbf{p}_1|} \int dE_{\mathbf{q}_1} |A_{fi}|^2 \delta(s - m_2'^2 + m_1'^2 - 2q_1\cdot(p_1 + p_2)).$$

This is simplest in the centre of mass frame. Let $p_1 = (\sqrt{|\mathbf{p}_1|^2 + m_1^2}, \mathbf{p}_1)$ and $p_2 = (\sqrt{|\mathbf{p}_1|^2 + m_2^2}, -\mathbf{p}_1)$. Then by considering s, we find:

$$|\mathbf{p}_1| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}, \quad \mathcal{F} = 2\lambda^{1/2}(s, m_1^2, m_2^2),$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Thus:

Differential cross section: In the centre of mass frame,

$$\left(\frac{d\sigma}{dt}\right)_{COM} = \frac{|A_{fi}|^2}{16\pi\lambda(s, m_1^2, m_2^2)}.$$

4.4 Decay rates

For a particle decaying, we are considering 1 to n scattering. Thus from above we need to consider:

$$\frac{dW}{d^4x} = \frac{|\psi(x)|^2}{2E_{\mathbf{p}}}|A_{fi}|^2(2\pi)^4\delta^4\left(p - \sum_i^n q_i\right).$$

This is equal to the probability density of the decaying particle multiplied by the rate at which it decays, $d\Gamma$, the *differential width*. Hence we have:

$$\Gamma = \frac{1}{2E_{\mathbf{p}}}\int\prod_{i=1}^n \frac{d^3\mathbf{q}_i}{(2\pi)^3 2E_{\mathbf{q}_i}}|A_{fi}|^2(2\pi)^4\delta^4\left(p - \sum_{i=1}^n q_i\right).$$

Note this is not Lorentz invariant; it is the time taken in the rest frame of the particle.

4.5 Example calculation

In scalar Yukawa theory, the decay width of a meson is:

$$\Gamma = \frac{g^2}{16\pi m}\left(1 - \frac{4\mu^2}{m^2}\right)^{1/2}.$$

To calculate this, we first find $|A_{fi}|^2 = g^2$ from the Feynman rules. Then we put this into the above formula, in the rest frame so that $p = (E_{\mathbf{p}}, 0) = (m, 0)$. The calculation also requires us to use the fact that

$$\delta(f(x)) = \sum_{\text{roots } x_i} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

5 The Dirac equation and spinors

5.1 The Lorentz algebra

Consider a column vector $\phi^a(x)$ of fields. Under a Lorentz transformation Λ, the most general transformation is:

$$\phi^a(x) = D^a{}_b(\Lambda)\phi^b(\Lambda^{-1}x).$$

Since applying two Lorentz transformations Λ_1, Λ_2 consecutively is the same as applying $\Lambda_2\Lambda_1$, we find that D is a *representation* of the Lorentz group. To get fermions in QFT we pick the *spinor representation*.

To find this rep, we look at the Lie algebra of the Lorentz group.

Definition: The Lie algebra of the Lorentz group is called the *Lorentz algebra*.

Write an infinitesimal Lorentz transformation as:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\omega^\mu{}_\nu + O(\epsilon^2).$$

We saw $\omega_{\mu\nu}$, a general element of the Lorentz algebra, is antisymmetric. Introduce a basis for the Lorentz algebra as (the obvious antisymmetric basis):

$$(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}.$$

Here, we have antisymmetry on $\rho\sigma$, since we need exactly 6 independent matrices to span. Lowering indices,

$$(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu.$$

We can then write a general Lorentz algebra element as

$$\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}\left(M^{\rho\sigma}\right)^\mu{}_\nu.$$

Here, $\Omega_{\rho\sigma}$ is antisymmetric (any symmetric part would cancel with $M^{\rho\sigma}$), and $\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu) = \Omega^\mu{}_\nu$.

Theorem: The structure constants of the basis are from:

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}.$$

Proof: Brief calculation. □

We can recover *finite* Lorentz transformations connected to the identity I by exponentiating: $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$.

5.2 The spinor representation

To construct the spinor representation of the Lorentz group (and of the Lorentz algebra) we go through a couple of stages.

Definition: The *Clifford algebra* is an algebra generated by objects γ^μ obeying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1$. Explicitly, this means $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\nu \neq \mu$ and $(\gamma^i)^2 = 1$, $(\gamma^0)^2 = -1$.

The simplest solution to the Clifford algebra is a set of 4×4 matrices.

Definition: The *chiral representation* of the Clifford algebra is the set of matrices

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli matrices, obeying their own algebra relations: $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k, \ \{\sigma^i, \sigma^j\} = 2\delta^{ij} I_2$.

Any similarity transformation $\gamma_\mu \mapsto U\gamma_\mu U^{-1}$ also gives a representation of the Clifford algebra.

Definition: The *spinor representation* of the Lorentz algebra is given by:

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}.$$

Theorem: $S^{\mu\nu}$ indeed constitutes a representation of the Lorentz algebra.

Proof: First show $[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\sigma\mu}$, then deduce

$$[S^{\rho\sigma}, S^{\tau\nu}] = \eta^{\sigma\tau}S^{\rho\nu} - \eta^{\rho\tau}S^{\sigma\nu} + \eta^{\sigma\nu}S^{\mu\tau} - \eta^{\sigma\nu}S^{\rho\tau}. \quad \square$$

Definition: The spinor representation of the Lorentz algebra gives rise to the *spinor representation* of the Lorentz group written:

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right).$$

Definition: A *Dirac spinor* is a collection of fields $\psi_\alpha(x)$, written as a column vector $(\psi_0(x), \psi_1(x), \psi_2(x), \psi_3(x))^T$, which transforms under a Lorentz transformation as:

$$\psi^\alpha(x) \mapsto S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x).$$

5.3 Rotations and boosts of spinors

The spinor representation is inequivalent to the fundamental (vector) representation of the Lorentz group. We can see this by considering rotations and boosts of spinors.

Theorem: Rotation of a spinor by 2π changes it by a minus sign; rotation by 4π leaves it invariant.

Proof: Use chiral representation of gamma matrices. For a rotation, the only non-zero $S^{\mu\nu}$ components are:

$$S^{ij} = \frac{1}{4}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\right] = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.$$

Since Ω_{ij} is antisymmetric, we can write it as $\Omega_{ij} = -\epsilon_{ijk}\phi^k$ for some vector $\boldsymbol{\phi}$. Then

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

For a 2π rotation, $\boldsymbol{\phi} = (0, 0, 2\pi)$. Then $S[\Lambda] = -I_4$. For a 4π rotation, $\boldsymbol{\phi} = (0, 0, 4\pi)$, then $S[\Lambda] = I_4$. $\square$

A vector would transform under a 2π rotation via:

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\pi & 0 \\ 0 & -2\pi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = I_4.$$

Theorem: Under a boost in the direction $\boldsymbol{\chi}$ by speed $|\boldsymbol{\chi}|$, the spinor representation of the Lorentz group is

$$S[\Lambda] = \begin{pmatrix} e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

Proof: Same as rotations. Only non-zero components are

$$S^{0i} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix},$$

and boost parameters may be written $\Omega_{0i} = -\Omega_{i0} = \chi_i$. $\square$

For boosts, $SS^\dagger = S^2$, so the representation is not unitary. In fact:

Theorem: There are no finite dimensional unitary representations of the Lorentz group.

Proof: Too hard for this course. $\square$

We can however prove that our spinor representation cannot be unitary:

Theorem: The spinor representation is not unitary, for any representation of the Clifford algebra.

Proof: The rep is unitary iff $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$. Note

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger].$$

So we would need all γ^μ's Hermitian or all γ^μ's anti-Hermitian. Since $(\gamma^0)^2 = I$, γ^0 has real eigenvalues so cannot be anti-Hermitian. Since $(\gamma^i)^2 = -I$, it cannot have real eigenvalues, so cannot be Hermitian. $\square$

5.4 Constructing a Lorentz invariant action

To build a Lorentz invariant action from spinors, need to make Lorentz scalars and Lorentz vectors from spinors.

Definition: The *Dirac adjoint* of $\psi(x)$ is $\overline{\psi}(x) := \psi^\dagger(x)\gamma^0 = (\psi^*)^T(x)\gamma^0$.

Lemma: $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0$

Proof: Work in the chiral rep of the Clifford algebra. Then $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$, so $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. Thus

$$(S^{\mu\nu})^\dagger = \frac{1}{4}\left[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\right] = -\gamma^0 S^{\mu\nu}\gamma^0. \quad \square$$

Lemma: $S[\Lambda]^{-1}\gamma^0 S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$.

Proof: We have $S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$. Hence

$$\begin{aligned} S[\Lambda]^{-1}\gamma^\mu S[\Lambda] &= \left(1 - \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 + \frac{1}{2}\Omega_{\tau\nu}S^{\tau\nu}\right) \\ &= \gamma^\mu - \frac{1}{2}\Omega_{\rho\sigma}[S^{\rho\sigma}, \gamma^\mu]. \end{aligned}$$

Compare to $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) = 1 + \frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}$. Thus, we must show $(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu = -[S^{\rho\sigma}, \gamma^\mu]$. Computing the LHS and RHS, we see they are indeed equal. $\square$

Theorem: $\overline{\psi}(x)\psi(x)$ is a Lorentz scalar, and $\overline{\psi}(x)\gamma^\mu\psi(x)$ is a Lorentz vector.

Proof: Under a Lorentz transformation $\overline{\psi}(x)\psi(x)$ maps to

$$\psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x) = \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x)$$

by the first Lemma. Under a Lorentz transformation, $\overline{\psi}(x)\gamma^\mu\psi(x)$ maps to

$$\overline{\psi}(\Lambda^{-1}x)S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi(\Lambda^{-1}x),$$

so apply second Lemma and we're done. $\square$

Definition: The *Dirac Lagrangian* is defined by

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x).$$

From above, this is Lorentz invariant (note $\partial_\mu \mapsto \Lambda^\nu{}_\mu\partial_\nu$ under a Lorentz transformation, i.e. transforms in opposite way to Lorentz vector).

This Lagrangian describes a free spinor field. The dimensions are: $[\psi] = 3/2$, and $[m] = 1$.

5.5 The Dirac equation

Theorem: The Euler-Lagrange equations of the Dirac action are:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \quad i(\partial_\mu\overline{\psi})\gamma^\mu + m\overline{\psi} = 0.$$

Proof: Varying $\overline{\psi}$, we get the first equation, and varying ψ we get the second equation (after integration by parts). $\square$

Definition: The equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ is called the *Dirac equation*.

Definition: We define $\not{A} = A_\mu\gamma^\mu = A^\mu\gamma_\mu$. This is called *slash notation*.

In slash notation, the Dirac equation is: $(i\not{\partial} - m)\psi = 0$.

The Dirac equation is, in a sense, the 'square root' of the Klein-Gordon equation. Indeed, each individual component of a spinor solves the Klein Gordon equation:

Theorem: Each spinor component solves the Klein-Gordon equation.

Proof: We have $(i\not{\partial} - m)\psi = 0$. Apply the operator $(i\not{\partial} + m)$. We then have:

$$\begin{aligned} 0 = (i\not{\partial} + m)(i\not{\partial} - m)\psi &= -(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi \\ &= -\left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2\right)\psi, \end{aligned}$$

since $\partial_\mu\partial_\nu$ is symmetric. Use Clifford and we're done. $\square$

5.6 Chiral spinors

$S[\Lambda]$ is block diagonal in the chiral rep. Hence S is the sum of irreps, acting on subspaces of the space of spinors.

Definition: We write a spinor in the chiral rep as $\psi = (u_L, u_R)^T$, where u_L and u_R are in $\mathbb{C}^2$. We call u_L and u_R *chiral* or *Weyl* spinors.

From the rotation and boost matrices above, u_L and u_R transform identically under rotations, but oppositely under boosts.

Decomposing the Dirac Lagrangian into Weyl spinors, we have:

$$\mathcal{L} = iu_L^\dagger\sigma^\mu\partial_\mu u_L + iu_R^\dagger\overline{\sigma}^\mu\partial_\mu u_R - m(u_L^\dagger u_R + u_R^\dagger u_L).$$

Here, we define $\sigma^\mu = (I, \boldsymbol{\sigma})$, $\overline{\sigma}^\mu = (I, -\boldsymbol{\sigma})$. Then the chiral rep is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}.$$

Terms like $u_L^\dagger u_R$ mean we annihilate a right Weyl spinor, and create a left Weyl spinor. This does not occur if $m = 0$, i.e. left/right handedness does not mix. However, it is impossible to prevent this occurring if $m \neq 0$.

Definition: In the massless case, the equations of motion reduce to

$$i\overline{\sigma}^\mu \partial_\mu u_L = 0, \quad i\sigma^\mu \partial_\mu u_R = 0,$$

which are called the *Weyl equations*.

5.7 Degrees of freedom

Definition: The *number of degrees of freedom* of a field theory is the half the dimension of the phase space at each spacetime point.

For a real scalar ϕ, the conjugate momentum is $\pi = \dot{\phi}$. So there is $\frac{1}{2} \times 2 = 1$ degree of freedom. For a complex scalar, there are two degrees of freedom (one for a particle, and one for the antiparticle).

For a spinor ψ, the conjugate momentum is $\pi = i\psi^\dagger$, which is *not* independent of ψ. So the 4 complex components of ψ give 8 real components, and *no more* from $i\psi^\dagger$. Hence there are $\frac{1}{2} \times 8$ degrees of freedom. These represent a *spin-*$\frac{1}{2}$ particle: spin-up particle, spin-down particle, spin-up antiparticle, spin-down antiparticle.

5.8 Rep independent Weyl spinors

Definition: Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Theorem: $\{\gamma^\mu, \gamma^5\} = 0$, and $(\gamma^5)^2 = I$.

Proof: $(\gamma^5)^2 = -(\gamma^0\gamma^1\gamma^2\gamma^3)(\gamma^0\gamma^1\gamma^2\gamma^3)$. Drag each gamma matrix one at a time through the string. Start with γ^0. It passes through three other gamma matrices, so pick up a minus sign, then hits γ^0 which gives $-I$. Repeat with others to get result.

The anti-commutation relation can be proved by observing that if Γ is a string of gamma matrices, then $\gamma^\mu\Gamma = (-1)^n\Gamma\gamma^\mu$ where n is the number of gamma matrices in Γ not equal to γ^μ (this follows since $\gamma^\mu\gamma^\nu = -\gamma^\mu\gamma^\nu$ for $\mu \neq \nu$ and $\gamma^\mu\gamma^\nu = \gamma^\nu\gamma^\mu$ if $\mu = \nu$). □

Definition: The *projection operators* are defined by $P_L = \frac{1}{2}(I - \gamma^5)$ and $P_R = \frac{1}{2}(I + \gamma^5)$.

Theorem: $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L P_R = 0$.

Proof: Simple calculation. □

Definition: Define a *left-handed spinor* by $\psi_L = P_L\psi$, where ψ is a Dirac spinor. Similarly define a *right-handed spinor* by $\psi_R = P_R\psi$.

We note that in the chiral rep,

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

so that P_R projects onto u_R and P_L projects onto u_L. However, the above Definition has allowed us to extend this in a rep-independent way.

5.9 Pseudoscalars, axial vectors and parity

Lemma: $[S_{\mu\nu}, \gamma^5] = 0$.

Proof: Use $S_{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ and $\{\gamma^5, \gamma^\mu\} = 0$. □

Using this Lemma, we see that $\overline{\psi}(x)\gamma^5\psi(x)$ is Lorentz invariant, and so is $\overline{\psi}(x)\gamma^5\gamma^\mu\psi(x)$. These are not scalars and vectors though! We will see why soon.

Definition: The *parity transformation* is denoted P. It transforms $x^0 \mapsto x^0$ and $x^i \mapsto -x^i$, i.e. it reflects space.

Theorem: Parity exchanges the left and right-handed chiral spinors: $Pu_L = u_R, Pu_L = u_R$.

Proof: We know $u_{L/R} \mapsto e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2}u_{L/R}$ under a rotation, and $u_{L/R} \mapsto e^{\pm i\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2}u_{L/R}$ under a boost. Parity does not affect rotations, but flips boosts. So indeed $Pu_{L/R} = u_{R/L}$. □

Immediately, this may be generalised to the rep-independent form: $P\psi_L = \psi_R$ and $P\psi_R = \psi_L$.

Theorem: For a Dirac spinor ψ, and the chiral rep of the Clifford algebra, parity acts as $P\psi = \gamma^0\psi$.

Proof: We note that in the chiral rep, $\psi = (u_L, u_R)^T$. Hence $P\psi = (u_R, u_L)^T = \gamma^0\psi$. □

This has an obvious generalisation to Dirac spinors not using the chiral rep.

Theorem: Under parity, $\overline{\psi}\psi$ is invariant. $\overline{\psi}\gamma^\mu\psi$ is invariant if $\mu = 0$, and changes sign if $\mu = i$.

Proof: Work in chiral rep. From above, $\overline{\psi}\psi \mapsto \psi^\dagger(\gamma^0)^\dagger(\gamma^0)^2\psi = \overline{\psi}\psi$, since $(\gamma^0)^\dagger = \gamma^0$. Similarly, get the result for vectors (get minus since $\{\gamma^i, \gamma^0\} = 0$). □

This is what we'd expect for a scalar and a vector.

However:

Theorem: Under parity, $\overline{\psi}\gamma^5\psi$ changes sign. $\overline{\psi}\gamma^5\gamma^\mu\psi$ also changes sign if $\mu = 0$, and is invariant if $\mu = i$.

Proof: Same proof as above. □

This is completely opposite to scalars and vectors!

Definition: We call these respective quantities *pseudoscalars* and *axial vectors*.

We can add pseudoscalar and axial vector terms to the Lagrangian. These terms actually arise in nature and break parity invariance, e.g. the weak force.

Definition: A theory which puts left and right-handed spinors on equal footing is called *vectorlike*. Else, a theory is called *chiral*.

5.10 Symmetries and conserved currents

Theorem: The energy-momentum tensor of the Dirac Lagrangian is

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi.$$

Proof: Under a translation $x^\mu \mapsto x^\mu - \epsilon^\mu$, the spinor transforms as $\psi(x) \mapsto \psi(x+\epsilon) = \psi(x) + \epsilon^\mu\partial_\mu\psi(x)$. Hence $T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}$.

To get the Noether current in general, we needed to impose the equations of motion. So impose $(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow \overline{\psi}(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow \mathcal{L} = 0$. □

Theorem: The Noether current corresponding to Lorentz transformations of the Dirac Lagrangian is

$$(J^\mu)^{\rho\sigma} = \underbrace{x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}}_{\text{orbital ang. momentum}} \underbrace{-i\overline{\psi}\gamma^\mu S^{\rho\sigma}\psi}_{\text{spin ang. momentum}}.$$

Proof: Under a Lorentz transformation, spinors transform as $\psi^\alpha \mapsto S[\Lambda]^\alpha{}_\beta\psi^\beta(\Lambda^{-1}x)$. Now recall $S[\Lambda]^\alpha{}_\beta = \delta^\alpha{}_\beta + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\alpha{}_\beta$, and $(\Lambda^{-1}x)^\mu = x^\mu - \omega^\mu{}_\nu x^\nu$. Hence the changes in ψ^α, $\overline{\psi}(\alpha)$ are (using $\Omega_{\mu\nu} = \omega_{\mu\nu}$ which we saw way back at the start of spinors):

$$\delta\psi^\alpha = -\omega^{\mu\nu}\left(x_\nu\partial_\mu\psi^\alpha - \frac{1}{2}(S_{\mu\nu})^\alpha{}_\beta\psi^\beta\right),$$
$$\delta\overline{\psi}_\alpha = -\omega^{\mu\nu}\left(x_\nu\partial_\mu\overline{\psi}_\alpha + \frac{1}{2}(S_{\mu\nu})^\beta{}_\alpha\overline{\psi}_\beta\right).$$

We've calculated $\delta\overline{\psi}_\alpha$ using $\overline{\psi} \mapsto \overline{\psi}S[\Lambda]^{-1}$ under the Lorentz transformation. To calculate the Noether current, use the standard formula and $\mathcal{L} = 0$. □

Theorem: The Dirac Lagrangian's internal symmetry $\psi \mapsto e^{i\alpha}\psi$ gives rise to the Noether current

$$j_V^\mu = \overline{\psi}\gamma^\mu\psi.$$

The conserved charge is $Q = \int d^3\mathbf{x}\psi^\dagger\psi$, which corresponds to charge/particle number conservation.

Proof: Standard method. □

Theorem: The Dirac Lagrangian's axial symmetry $\psi \mapsto e^{i\alpha\gamma^5}\psi$, which appears only in the massless limit $m = 0$, gives rise to the conserved current

$$j_A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi.$$

Proof: Note $e^{-i\alpha\gamma^5}\gamma^0 = \gamma^0 e^{i\alpha\gamma^5}$. So $\overline{\psi} \mapsto \overline{\psi}e^{i\alpha\gamma^5}$ under this transformation. Now applying the standard method gives the current. □

Axial symmetry is interesting. This is because it is a symmetry that does *not* hold after quantisation. It is called an *anomaly*.

5.11 Plane-wave solutions of Dirac equation

To quantise, we need to study plane-wave solutions of the Dirac equation.

Theorem: A solution of the Dirac equation of the form $\psi = u(\mathbf{p})e^{-ip\cdot x}$ is given by:

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix},$$

where ξ is a two component spinor, which we can normalise such that $\xi^\dagger\xi = 1$.

Proof: Substitute into the Dirac equation in the chiral rep to obtain

$$(p\!\!\!/ - mI)u(\mathbf{p}) = 0 \quad \Rightarrow \quad \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\overline{\sigma}^\mu & -m \end{pmatrix} u(\mathbf{p}) = 0.$$

Insert $u(\mathbf{p}) = (u_1, u_2)^T$. Then $(p\cdot\sigma)u_2 = mu_1$ and $(p\cdot\overline{\sigma})u_1 = mu_2$. Each equation implies the other, since $(p\cdot\sigma)(p\cdot\overline{\sigma}) = ... = p_\mu p^\mu = m^2$. So $(p\cdot\overline{\sigma})(p\cdot\sigma) = (p\cdot\sigma)(p\cdot\overline{\sigma})$.

WLOG, write $u_1 = (p\cdot\sigma)\xi^1$. Then the second equation implies $u_2 = m\xi^1$. Thus we find

$$u(\mathbf{p}) = A\begin{pmatrix} (p\cdot\sigma)\xi^1 \\ m\xi^1 \end{pmatrix}$$

is a solution. Choosing the normalisation constant $A = 1/m$ and choosing $\xi = \sqrt{p\cdot\overline{\sigma}}\xi^1$, WLOG, we get the given solution. □

It is also possible to get negative frequency solutions: $\psi = v(\mathbf{p})e^{ip\cdot x}$, which take the form:

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta \\ -\sqrt{p\cdot\overline{\sigma}}\eta \end{pmatrix},$$

where $\eta^\dagger\eta = 1$.

Note: ξ and η describe the *spin* of the field. $\xi = (1,0)$ corresponds to spin up and $\xi = (0,1)$ corresponds to spin down.

Definition: The *helicity* operator is defined by $h = \hat{\mathbf{p}}\cdot\mathbf{s}$, i.e. the projection of angular momentum along the direction of motion:

$$h = \frac{i}{2}\epsilon_{ijk}\hat{p}^i S^{jk} = \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

5.12 Properties of the plane-wave solutions

Theorem: In a basis of orthonormal 2-component spinors ξ^s, $s = 1,2$, such that $(\xi^\dagger)^r\xi^s = \delta^{rs}$, we have the orthogonality relations:

$$u^r(\mathbf{p})^\dagger\cdot u^s(\mathbf{p}) = 2p_0\delta^{rs}, \quad \overline{u}^r(\mathbf{p})\cdot u^s(\mathbf{p}) = 2m\delta^{rs},$$
$$v^r(\mathbf{p})^\dagger\cdot v^s(\mathbf{p}) = 2p_0\delta^{rs}, \quad \overline{v}^r(\mathbf{p})\cdot v^s(\mathbf{p}) = -2m\delta^{rs},$$
$$\overline{u}^s(\mathbf{p})\cdot v^r(\mathbf{p}) = 0, \quad u^s(\mathbf{p})^\dagger\cdot v^r(-\mathbf{p}) = 0.$$

Proof: Note $(\sqrt{p\cdot\sigma})^2 = p\cdot\sigma \Rightarrow ((\sqrt{p\cdot\sigma})^\dagger)^2 = p\cdot\sigma^\dagger = p\cdot\sigma$, so assuming we take a canonical square root, $(\sqrt{p\cdot\sigma})^\dagger = \sqrt{p\cdot\sigma}$. Similarly $(\sqrt{p\cdot\overline{\sigma}})^\dagger = \sqrt{p\cdot\overline{\sigma}}$.

The formulae then follow from a natural calculation, using the facts that $p\cdot\sigma + p\cdot\overline{\sigma} = 2p_0 I$, and $(p\cdot\sigma)(p\cdot\overline{\sigma}) = m^2$. For the very last relation, it's easiest to write $p' = (p_0, -\mathbf{p})$.$\square$

Theorem: The following formulae hold for the outer-products of the plane-wave solutions:

$$\sum_{s=1}^{2} u^s(\mathbf{p})\overline{u}^s(\mathbf{p}) = \not{p} + m, \quad \sum_{s=1}^{2} v^s(\mathbf{p})\overline{v}^s(\mathbf{p}) = \not{p} - m.$$

Proof: Follow similar calculations as in the previous proof. This time we need to know:

$$\sum_{s=1}^{2}\xi^s(\xi^s)^\dagger.$$

This can quickly be calculated to be the identity using an orthonormal basis for the 2-component spinors: $\xi^1 = (1,0)$ and $\xi^2 = (0,1)$. This gives:

$$\sum_{s=1}^{2}\xi^s(\xi^s)^\dagger = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = I.$$

Using this gives the result, together with some calculations. $\square$

5.13 Trace theorems

When calculating cross-sections and decay rates for spinor fields, we need to use trace theorems of the form:

Theorem (Examples): (i) The trace of an odd number of gamma matrices is zero (none of them γ^5); (ii) $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$; (iii) $\mathrm{tr}(\gamma^5) = 0$; (iv) $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$; (v) $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}$.

Proof: (i) Consider $\mathrm{tr}(\gamma_1^{\mu_1}\gamma_2^{\mu_2}...\gamma_{2n+1}^{\mu_{2n+1}})$. We can insert a $(\gamma^5)^2 = I$, and use the following argument:

$$\begin{aligned}\mathrm{tr}(\gamma_1^{\mu_1}\gamma_2^{\mu_2}...\gamma_{2n+1}^{\mu_{2n+1}}) &= \mathrm{tr}(\gamma_1^{\mu_1}\gamma_2^{\mu_2}...\gamma_{2n+1}^{\mu_{2n+1}}(\gamma^5)^2) \\ &= \mathrm{tr}(\gamma^5\gamma_1^{\mu_1}\gamma_2^{\mu_2}...\gamma_{2n+1}^{\mu_{2n+1}}\gamma^5) \qquad \text{(cyclicity)} \\ &= (-1)^{2n+1}\mathrm{tr}(\gamma_1^{\mu_1}\gamma_2^{\mu_2}...\gamma_{2n+1}^{\mu_{2n+1}}),\end{aligned}$$

by anticommuting γ^5 back through all other gamma matrices (since $\{\gamma^5,\gamma^\mu\} = 0$). Hence trace is zero.

(ii) To find the trace of an even number of gamma matrices, we use a similar trick. Use cyclicity to move one gamma matrix to the other end of the string, then anti-commute it back to the beginning:

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = \mathrm{tr}(\gamma^\nu\gamma^\mu) = \mathrm{tr}(-\gamma^\mu\gamma^\nu + 2\eta^{\mu\nu}I),$$

and so the result follows (use $\mathrm{tr}(I) = 4$).

(iii) Use same trick as (i), but insert $(\gamma^0)^2 = I$ instead of $(\gamma^5)^2 = I$.

(iv) Pick $\alpha \neq \mu,\nu$ and insert $\gamma^{\alpha 2}$, then same trick as (iii) works.

(v) We note that interchanging any two of the gamma matrices in the trace changes the sign of the answer. So this trace must be proportional to $\epsilon^{\mu\nu\rho\sigma}$. Hence $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \lambda\epsilon^{\mu\nu\rho\sigma}$. Picking $\mu\nu\rho\sigma = 0123$, and recalling from general relativity that $\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}$ in Minkowski spacetime, we have

$$-\lambda = \mathrm{tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) = -i\mathrm{tr}(\gamma^{5^2}) = -4i. \quad \square$$

6 Quantising the Dirac field

6.1 Anti-commutation relations

In the Schrödinger picture, the quantised spinor fields are:

$$\psi(\mathbf{x}) = \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(b^s_\mathbf{p} u^s(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + (c^s_\mathbf{p})^\dagger v^s(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \right),$$

$$\psi^\dagger(\mathbf{x}) = \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left((b^s_\mathbf{p})^\dagger u^s(\mathbf{p})^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + c^s_\mathbf{p} v^s(\mathbf{p})^\dagger e^{i\mathbf{p}\cdot\mathbf{x}} \right)$$

Here, the sum is over possible spins, the annihilation and creation operators are introduced for *both* spin states, and u and v are the positive and negative frequency plane-wave solutions to the Dirac equation. Note that both these objects have four components, i.e. we've omitted indices: $\psi(\mathbf{x}) \equiv \psi_\alpha(\mathbf{x})$.

Unlike the KG field we saw earlier, we need to impose *anticommutation relations* on the spinor fields:

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = 0 = \{\psi^\dagger_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{x})\},$$
$$\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}).$$

Theorem: These anticommutation relations imply $\{b^r_\mathbf{p}, (b^s_\mathbf{q})^\dagger\} = (2\pi)^3\delta^{rs}\delta^3(\mathbf{p}-\mathbf{q})$, and $\{c^r_\mathbf{p}, (c^s_\mathbf{q})^\dagger\} = (2\pi)^3\delta^{rs}\delta^3(\mathbf{p}-\mathbf{q})$, with all other anticommutators zero.

Proof: Same as in bosonic case. However, we do need to use some spinor identities from the previous section. Best to include all indices.

Also note that to kill off the gamma matrices, we need to consider $\mathbf{p} \mapsto -\mathbf{p}$, parity, etc. □

If we instead assumed commutation relations on the spinor fields, we would find that $[b^r_\mathbf{p}, (b^s_\mathbf{q})^\dagger] = (2\pi)^3\delta^{rs}\delta^3(\mathbf{p}-\mathbf{q})$, and $[c^r_\mathbf{p}, (c^s_\mathbf{q})^\dagger] = -(2\pi)^3\delta^{rs}\delta^3(\mathbf{p}-\mathbf{q})$. The weird minus sign means that we must interpret c as the creation operator and $c^\dagger$ as the annihilation operator. This leads to the Hamiltonian being unbounded below, which is unphysical.

6.2 The Hamiltonian

Theorem: The classical Hamiltonian for a spinor field is

$$\mathcal{H} = \overline{\psi}(-i\gamma^i\partial_i + m)\psi.$$

Proof: The conjugate momentum is $\pi = i\psi^\dagger$. So simply computing $\mathcal{H} = \pi\dot{\psi} - \mathcal{L}$, we get the result. Note all indices have been suppressed. □

Theorem: The quantised, normal-ordered Hamiltonian is:

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_\mathbf{p} \sum_{s=1}^{2} \left((b^s_\mathbf{p})^\dagger b^s_\mathbf{p} + (c^s_\mathbf{p})^\dagger c^s_\mathbf{p} \right).$$

Proof: The trick is to evaluate it in chunks. Since $\mathbf{p}\cdot\mathbf{x} = -x^i p_i$, $\partial_i e^{i\mathbf{p}\cdot\mathbf{x}} = -ip_i e^{i\mathbf{p}\cdot\mathbf{x}}$. Hence $(-i\gamma^i\partial_i + m)\psi_\alpha =$

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{s=1}^{2} \Big(b^s_\mathbf{p}(-\gamma^i p_i + m)_{\alpha\beta} u^s_\beta(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + (c^s_\mathbf{p})^\dagger \left(-\gamma^i p_i + m\right)_{\alpha\beta} v^s_\beta(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \Big).$$

Recall that $u^s(\mathbf{p})$ and $v^s(\mathbf{p})$ are solutions of the Dirac equation with $(\gamma^\mu p_\mu - m)u^s(\mathbf{p}) = 0$ and $(\gamma^\mu p_\mu + m)v^s(\mathbf{p})$. Hence we can rewrite the above as:

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \sqrt{\frac{E_\mathbf{p}}{2}} \gamma^0_{\alpha\beta} \sum_{s=1}^{2} \Big(b^s_\mathbf{p} u^s_\beta(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + (c^s_\mathbf{p})^\dagger v^s_\beta(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \Big).$$

The rest of the calculation is relatively straightforward, but we do need to use the inner-product formulae for the plane-wave spinors partway through the calculation. □

Theorem: $[H, (b^r_\mathbf{p})^\dagger] = E_\mathbf{p}(b^r_\mathbf{p})^\dagger$, $[H, b^r_\mathbf{p}] = -E_\mathbf{p} b^r_\mathbf{p}$. Similar relations hold for the c's.

Proof: Trivial from above. □

This shows that we can interpret b and c as creation and annihilation operators.

We now label particles by their spin as well as their momentum: $|\mathbf{p}, r\rangle := (b^r_\mathbf{p})^\dagger |0\rangle$. We note that because we have anti-commutation relations, $|\mathbf{p}_1, r_1; \mathbf{p}_2, r_2\rangle = -|\mathbf{p}_2, r_2; \mathbf{p}_1, r_1\rangle$. That is, these particles are *fermions*.

6.3 Heisenberg fields

As in the bosonic case, the Heisenberg fields are:

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(b^s_\mathbf{p} u^s(\mathbf{p}) e^{-ip\cdot x} + (c^s_\mathbf{p})^\dagger v^s(\mathbf{p}) e^{ip\cdot x} \right),$$

$$\psi^\dagger(x) = \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left((b^s_\mathbf{p})^\dagger u^s(\mathbf{p})^\dagger e^{ip\cdot x} + c^s_\mathbf{p} v^s(\mathbf{p})^\dagger e^{-ip\cdot x} \right)$$

6.4 Causality and propagators

As in bosonic theory, we want to study the causality of the theory.

Definition: Define (in analogy to $\Delta(x-y)$ in bosonic theory) the function:

$$iS_{\alpha\beta}(x-y) = \{\psi_\alpha(x), \overline{\psi}_\beta(y)\}.$$

Theorem: $iS(x-y) = (i\partial\!\!\!/_x + m)[D(x-y) - D(y-x)]$, where $\partial\!\!\!/_x = \gamma^\mu \partial/\partial x^\mu$, and $D(x-y)$ is the propagator as previously defined.

Proof: Via a short calculation. Need to use outer product identity for plane-wave spinors. □

We know that for spacelike separated x and y, $D(x-y) - D(y-x) = 0$. Hence S vanishes for spacelike separations, i.e. the anticommutator $\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = 0$ for spacelike separations.

This appears wrong! But it is not. In fermionic theory, all observables are bilinear in $\overline{\psi}$ and ψ, hence satisfy normal commutation relations, and hence do commute at spacelike separations as a consequence of the above.

6.5 The Feynman propagator

Definition: The *Feynman propagator* of fermionic theory is defined by:

$$S_F(x-y) = \langle 0|T\{\psi(x)\overline{\psi}(y)\}|0\rangle .$$

Note this is a 4×4 matrix. Also, the time-ordering operator is defined differently for fermions:

$$T\{\psi(x)\overline{\psi}(y)\} = \begin{cases} \psi(x)\overline{\psi}(y) \text{ if } x^0 > y^0, \\ -\overline{\psi}(y)\psi(x) \text{ otherwise.} \end{cases}$$

The reason the minus is required is for Lorentz invariance. When we have spacelike separated x and y; both time-orderings are acceptable depending on the frame, and we need to impose $\{\psi(x), \overline{\psi}(y)\} = 0$ regardless of frame.

Slogan: Strings of fermionic operators inside a time-ordering *anticommute*.

Theorem: The Feynman propagator has integral representation:

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}.$$

Proof: Completely analogous to bosonic case. □

Theorem: $S_F(x-y)$ is a Green's function for the Dirac equation.

Proof: Again, analogous to bosonic case. □

6.6 Changes to Wick's Theorem

Normal ordering requires modification in fermionic theory; it is no longer symmetric under interchange of fields. We find:

$$:\psi_1\psi_2: = -:\psi_2\psi_1: .$$

Contractions are defined similar to before: $\overbracket{\psi(x)\overline{\psi}(y)} = S_F(x-y)$, with all other fermionic contractions zero (including contractions of a spinor field with a scalar field).

Wick's Theorem is affected because we can only contract fields which are next door now, though:

$$:\contraction{}{\psi_1}{\psi_2}{\overline{\psi}_3}\psi_1\psi_2\overline{\psi}_3\psi_4: = -:\contraction{}{\psi_1}{}{\overline{\psi}_3}\psi_1\overline{\psi}_3\psi_2\psi_4: = -S_F(x_1-x_3):\psi_2\psi_4: .$$

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7 Interacting fermionic theory

7.1 Fermionic Yukawa theory

Definition: The Lagrangian of *fermionic Yukawa theory* is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \lambda\phi\bar{\psi}\psi.$$

Analysing dimensions, $[\phi] = 1$, $[\psi] = \frac{3}{2}$, so $[\lambda] = 0$. Hence this is a renormalisable theory.

Example: Consider nucleon-nucleon scattering $\psi(p,s)\psi(q,r) \to \psi(p',s')\psi(q',r')$ (where s, r, s' and r' label the spins). The initial and final states are:

$$|i\rangle = \sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{q}}}\, b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle , \quad |f\rangle = \sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{q}}}\, b_{\mathbf{p}'}^{s'\,\dagger} b_{\mathbf{q}'}^{r'\,\dagger}|0\rangle$$

Thinking about the interaction term $\bar{\psi}\psi\phi$, we see that the first non-zero contribution to the scattering is (from Dyson's formula): $\langle f|(S-1)|i\rangle =$

$$\langle f|\,\frac{(-i\lambda)^2}{2!}\int d^4x_1 d^4x_2\, T\{\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\bar{\psi}(x_2)\psi(x_2)\phi(x_2)\}\,|i\rangle .$$

Wick's Theorem allows us to expand the time-ordering. We must contract the scalar fields, since if left uncontracted, normal ordering would force them to annihilate $|i\rangle$ or $|f\rangle$. This leaves us with the only contributing term:

$$\langle f|\,\frac{(-i\lambda)^2}{2!}\int d^4x_1 d^4x_2\ :\bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2):\Delta_F(x_1-x_2)\,|i\rangle .$$

Deal with the action of the normal ordering on $|i\rangle$ first. Note that the creation operators $b^\dagger$ must be cancelled by the annihilation operators b in the two ψ's. So anticommuting inside the normal ordering, we have (showing spinor indices explicitly):

$$:\bar{\psi}_\alpha(x_1)\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\psi_\beta(x_2): b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle =$$
$$- :\bar{\psi}_\alpha(x_1)\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)\psi_\beta(x_2): b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle =$$

$$-\int \frac{d^3\mathbf{k}_1\, d^3\mathbf{k}_2}{(2\pi)^6 2\sqrt{E_{\mathbf{k}_1}E_{\mathbf{k}_2}}}[\bar{\psi}_\alpha(x_1)u^m_{\mathbf{k}_1,\alpha}][\bar{\psi}_\beta(x_2)u^n_{\mathbf{k}_2,\beta}]\cdot$$
$$e^{-i(k_1\cdot x_1+k_2\cdot x_2)} b^m_{\mathbf{k}_1} b^n_{\mathbf{k}_2} b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle\,,$$

where all other terms in the mode expansion of $\psi(x_1)$, $\psi(x_2)$ cancel, since they contain c's and $c^\dagger$'s, which annihilate $|i\rangle$ or $|f\rangle$. Simplifying the b, $b^\dagger$ expression using the anticommutation relations, we find that

$$b^m_{\mathbf{k}_1} b^n_{\mathbf{k}_2} b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle = (2\pi)^6(\delta^3(\mathbf{k}_2-\mathbf{p})\delta^3(\mathbf{k}_1-\mathbf{q})\delta^{ns}\delta^{mr}$$
$$-\delta^3(\mathbf{k}_1-\mathbf{p})\delta^3(\mathbf{k}_2-\mathbf{p})\delta^{ms}\delta^{nr})\,|0\rangle\,.$$

This gives the final expression for the $|i\rangle$ side:

$$:\bar{\psi}_\alpha(x_1)\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\psi_\beta(x_2): b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle =$$
$$-\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\Big([\bar{\psi}(x_1)u^r_{\mathbf{q}}][\bar{\psi}(x_2)u^s_{\mathbf{p}}]e^{-i(q\cdot x_1+p\cdot x_2)}$$
$$-[\bar{\psi}(x_1)u^s_{\mathbf{p}}][\bar{\psi}(x_2)u^r_{\mathbf{q}}]e^{-i(p\cdot x_1+q\cdot x_2)}\Big)\,|0\rangle\,.$$

Now applying this to $|f\rangle$ on the left (careful when calculating $|f\rangle$, because the order of the creation operators matters - they anticommute!), and expanding $\bar{\psi}$ in exactly the same way, we find that $\langle f|(S-1)|i\rangle =$

$$-\frac{(-i\lambda)^2}{2!}\int d^4x_1 d^4x_2\Big([\bar{u}^{s'}_{\mathbf{p}'}u^r_{\mathbf{q}}][\bar{u}^{r'}_{\mathbf{q}'}u^s_{\mathbf{p}}]\; e^{ix_1\cdot(p'-q)+ix_2\cdot(q'-p)}$$
$$-(r'\leftrightarrow s', p'\leftrightarrow q')\Big)\Delta_F(x_1-x_2).$$

Use the expression for the scalar Feynman propagator:

$$\Delta_F(x_1-x_2) = \int\frac{d^4\mathbf{k}}{(2\pi)^4}\frac{ie^{-ik\cdot(x_1-x_2)}}{k^2-\mu^2+i\epsilon}$$

to convert the exponentials into delta functions. This gives the final result:

$$A_{fi} = -(-i\lambda)^2\left(\frac{[\bar{u}^{s'}_{\mathbf{p}'}u^r_{\mathbf{q}}][\bar{u}^{r'}_{\mathbf{q}'}u^s_{\mathbf{p}}]}{(q'-p)^2-\mu^2+i\epsilon} - \frac{[\bar{u}^{r'}_{\mathbf{q}'}u^r_{\mathbf{q}}][\bar{u}^{s'}_{\mathbf{p}'}u^s_{\mathbf{p}}]}{(p'-p)^2-\mu^2+i\epsilon}\right).$$

Note that the overall sign of the answer is up for debate - for example, we could have chosen $|i\rangle = \sqrt{2E_{\mathbf{q}}}\sqrt{2E_{\mathbf{p}}}b_{\mathbf{q}}^{r\,\dagger}b_{\mathbf{p}}^{s\,\dagger}|0\rangle$, which reproduces this result only by anticommuting the $b^\dagger$'s before we start.

However, this doesn't matter, since for all observables (i.e. cross-sections and decay rates), we take the modulus squared of A_{fi} first.

7.2 Feynman rules for fermions

Calculations such as these lead to:

The Feynman rules: The amplitude A_{fi} (up to a minus sign) is given by the following procedure. Begin by drawing all possible Feynman diagrams for the process. To each diagram, associate a value via:

1. At every vertex, write down a factor of $(-i\lambda)$ (obviously this is different for different interaction terms).
2. Impose 4-momentum conservation at every vertex.
3. For each internal scalar ϕ line, write a factor of the scalar propagator:
$$\frac{i}{p^2-\mu^2-i\epsilon}.$$
4. For each incoming fermion line, write a $u^s_{\mathbf{p}}$, and for each outgoing fermion write a $\bar{u}^s_{\mathbf{p}}$.
5. For each incoming anti-fermion, write a $v^s_{\mathbf{p}}$ and for each outgoing anti-fermion write a $\bar{v}^s_{\mathbf{p}}$.
6. For each internal fermion line, with spinor indices $\alpha\to\beta$, write a fermion propagator:
$$\frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}.$$
7. Contract spinor indices meeting at a vertex (in practice, it's easiest to work back to front in the diagram when doing this).
8. For each closed fermionic loop, introduce an additional minus sign.
9. Integrate over all undetermined momenta.

Rule 8 may seem a little mysterious, but ultimately comes from the anti-commutativity of fermion fields. For example, for $\phi(p)\to\phi(p)$ propagation, we can include the one-loop term:

Wick's Theorem requires we contract all fermion fields as:

$$:\bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y):$$

(Note we can't contract two α's, because when they are next to one another, they are just a number!) To get the standard form of the spinor contraction, we must anticommute the final ψ to the front, past three spinor fields. So we get an extra factor of -1, as per the Feynman rule.

7.3 Cross-sections & decays of fermions

Let's now make some predictions using the theory. First of all, we make some modifications due to spin.

Definition: In most experiments, beams of particles are prepared with random initial spin; therefore, we *average* over initial spins when we calculate cross-sections and decay rates. Also, unless we want to discriminate particular spins for sum reasons, we sum over all the possible final spin states. Thus the final cross-section/decay rate should use the modified amplitude squared:

$$\frac{1}{4}\sum_{r',s',r,s}|A_{fi}|^2.$$

This procedure is called taking the *spin-sum average*. We write $\overline{|A_{fi}|^2}$ for the spin-sum average of $|A_{fi}|^2$.

Example: For nucleon-nucleon scattering as above, we found that A_{fi} was of the form $A_{fi} = A - B$, for two horrible expressions A and B. Therefore,

$$\overline{|A_{fi}|^2} = \overline{|A|^2} + \overline{|B|^2} - \overline{A^\dagger B} - \overline{B^\dagger A}.$$

Let's calculate $\overline{|A|^2}$ here, and do the rest later. Recall that

$$A = \frac{\lambda^2}{u-\mu^2+i\epsilon}[\bar{u}^{s'}_{\mathbf{p}'}u^r_{\mathbf{q}}][\bar{u}^{r'}_{\mathbf{q}'}u^s_{\mathbf{p}}],$$

where u is one of the Mandelstam variables. So, $\overline{|A|^2} =$

$$\frac{\lambda^4}{4(u-\mu^2)^2}\sum_{r,s,r',s'}\bar{u}^{s'}_{\mathbf{p}',\alpha}\underbrace{u^r_{\mathbf{q},\alpha}\bar{u}^r_{\mathbf{q},\beta}}_{=(\not{q}+m)_{\alpha\beta}}u^{s'}_{\mathbf{p}',\beta}\bar{u}^{r'}_{\mathbf{q}',\gamma}\underbrace{u^s_{\mathbf{p},\gamma}\bar{u}^s_{\mathbf{p},\delta}}_{=(\not{p}+m)_{\gamma\delta}}u^{r'}_{\mathbf{q}',\delta},$$

where the underbraces show how we can use the outer product identities from when we studied spinors a long time ago. Thus we get:

$$\frac{\lambda^4}{4(u-\mu^2)^2}\mathrm{Tr}((\not{q}+m)(\not{p}'+m))\mathrm{Tr}((\not{p}+m)(\not{q}'+m)).$$

It's now possible to use the trace identities from way back when we studied spinors to calculate:

$$\overline{|A|^2} = \frac{4\lambda^4}{(u-\mu^2)^2}(q\cdot p'+m^2)(p\cdot q'+m^2)$$

We can write $q\cdot p'$ and $p\cdot q'$ in terms of the Mandelstam variables via:

$$u = (p-q')^2 = p\cdot p - 2p\cdot q' + q'\cdot q' = 2m^2 - 2p\cdot q',$$

so $p\cdot q' = m^2 - \frac{1}{2}u$. Similarly, $p'\cdot q = m^2 - \frac{1}{2}u$. Hence:

$$\overline{|A|^2} = \frac{(u-4m^2)^2}{(u-\mu^2)^2}.$$

7.4 Diagrammatic calculation of $\overline{|A_{fi}|^2}$

Given generic terms C, D in A_{fi}, suppose we want to compute $\overline{CD^\dagger}$. There is a recipe in terms of diagrams to go straight to the trace form of the answer.

1. Draw the Feynman diagram that gave C, and draw the Feynman diagram for D next to it, but with *initial and final momenta exchanged*.

2. Join up all fermion lines with identical momenta.

3. Apply the Feynman rules, as usual. The rule for a fermion loop in this case, however, is to take the trace of the product of all matrices $(\not{p}+m)$ where p is the momentum of each fermion in the loop. We follow the loop backwards when multiplying the matrices.

Example: We can calculate A in nucleon-nucleon scattering using the method above. The A and $A^\dagger$ Feynman diagrams side by side are:

So inserting the loop we have:

This immediately gives the trace result we got before, by the Feynman rules and the fermion loop rule.

Example: We can finally calculate the whole cross section for nucleon-nucleon scattering now. Using the above techniques we can compute $\overline{|A|^2}$, $\overline{|B|^2}$ and $\overline{AB^\dagger}$ (note that $AB^\dagger + BA^\dagger = 2\,\mathrm{Re}(AB^\dagger)$, so it's sufficient just to compute this). Putting them all into the formula for 2 to 2 scattering, we find:

$$\frac{d\sigma}{dt} = \frac{\lambda^4}{16\pi s(s-4m^2)^2}\left(\frac{(u-4m^2)^2}{(u-\mu^2)^2} + \frac{(t-4m^2)^2}{(t-\mu^2)^2} + \frac{1}{2}\left(\frac{(s-4m^2)^2-(u-4m^2)^2-(t-4m^2)^2}{(t-\mu^2)(u-\mu^2)}\right)\right)$$

To get the full cross section from the differential cross section, we need to integrate over t.

Recall that in the centre of mass frame, the sum of three momentum is zero, so $\mathbf{p} = -\mathbf{q}$, $\mathbf{p}' = -\mathbf{q}'$, and by conservation of energy,

$$\sqrt{m^2+|\mathbf{p}|^2}+\sqrt{m^2+|\mathbf{q}|^2}=\sqrt{m^2+|\mathbf{p}'|^2}+\sqrt{m^2+|\mathbf{q}'|^2}$$
$$\Rightarrow |\mathbf{p}|=|\mathbf{p}'|.$$

Thus all energies and masses are equal in this problem. Hence $t=(p-p')^2$

$$=p^2+{p'}^2-2p\cdot p'=E^2-|\mathbf{p}|^2+{E'}^2-|\mathbf{p}'|^2-2EE'+2\mathbf{p}\cdot\mathbf{p}'$$
$$=2|\mathbf{p}|^2(\cos(\theta)-1)$$

Since $\theta\in[0,\pi]$ is the scattering angle, it follows we must integrate over the range $[-4|\mathbf{p}|^2,0]$.

We can finish the whole calculation in the massless limit. Recall that:

$$s=(p+q)^2=p^2+q^2+2p\cdot q=2m^2+2(E_{\mathbf{p}}E_{\mathbf{q}}-\mathbf{p}\cdot\mathbf{q})$$
$$=2m^2+2(m^2+|\mathbf{p}|^2+|\mathbf{p}|^2)=4(m^2+|\mathbf{p}|^2).$$

Note we have $dt=2|\mathbf{p}|^2d\cos(\theta)$, and so in the massless limit

$$\frac{dt}{d\cos(\theta)}=\frac{s}{2}.$$

Hence:

$$\frac{d\sigma}{d\Omega}=\frac{s}{4\pi}\frac{d\sigma}{dt}=\frac{3\lambda^4}{64\pi^2 s}.$$

We can now integrate over the sphere, since s has no θ or ϕ dependence. We get:

$$\sigma=\frac{3\lambda^4}{16\pi s}.$$

This is actually wrong by a factor of 2 - why? Because the final particles are identical, in the big cross-section formula where we integrate over momenta, in some regions we'll count the same scenario twice. Thus we must divide by 2 right at the end of the calculation.

8 Quantum electrodynamics

8.1 Definitions and gauge invariance

Definition: The *photon field* is a vector field written A_μ. The *field-strength tensor* for A_μ is given by:

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu.$$

The Lagrangian for the *free electrodynamic theory* is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Theorem: The equation of motion of A_μ is $\partial_\mu F^{\mu\nu}=0$.

Proof: Quick calculation. □

Theorem: The *Bianchi identity* holds:

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0.$$

Proof: Just insert definition in terms of A_μ and check. □

The above definitions and theorems are motivated by the following. When we write:

$$A^\mu=\begin{pmatrix}\phi\\ \mathbf{A}\end{pmatrix},$$

and define the electric and magnetic fields by:

$$\mathbf{E}=-\nabla\phi-\dot{\mathbf{A}},\quad \mathbf{B}=\nabla\times\mathbf{A},$$

then the field-strength tensor becomes:

$$F_{\mu\nu}=\begin{pmatrix}0&E_1&E_2&E_3\\-E_1&0&-B_3&B_2\\-E_2&B_3&0&-B_1\\-E_3&-B_2&B_1&0\end{pmatrix}.$$

When inserted into $\partial_\mu F^{\mu\nu}=0$, this recovers the unsourced Maxwell equations: $\nabla\cdot\mathbf{E}=0$ and $\dot{\mathbf{E}}=\nabla\times\mathbf{B}$. When inserted into the Bianchi identity, we recover the Maxwell equations $\nabla\cdot\mathbf{B}=0$ and $\dot{\mathbf{B}}=-\nabla\times\mathbf{E}$.

We now want to quantise. We know from our general physics knowledge that the photon has two real degrees of freedom, but A_μ has 4! How do we cut these down? We notice the following about this theory:

- A_0 is time independent, since it has no kinetic term in the Lagrangian (would need $\partial_0 A_0$ - not allowed by antisymmetry of $F_{\mu\nu}$).

  Also note that $\nabla\cdot\mathbf{E}=0$ implies $\nabla^2 A_0+\nabla\cdot\dot{\mathbf{A}}=0$, which can be solved via Green's function:

  $$A_0=\int d^3\mathbf{x}'\,\frac{\nabla\cdot\dot{\mathbf{A}}}{4\pi|\mathbf{x}-\mathbf{x}'|},$$

  so A_0 is *completely determined* by the evolution of the other fields. Thus reduced to 3 degrees of freedom.

- The Lagrangian has a *gauge symmetry* given by $A_\mu \mapsto A_\mu + \partial_\mu \lambda(x)$, where λ is any function such that $\lambda \to 0$ as $|\mathbf{x}| \to \infty$. It's easy to check $F_{\mu\nu}$ is invariant under this symmetry, and thus so is $\mathcal{L}$.

Gauge symmetries are not like true symmetries. They represent a *redundancy in our description of the system* in the following sense. The equation of motion can be written as $\eta_{\mu\nu}\partial_\rho F^{\rho\nu} = 0$, so expanding $F^{\rho\nu}$, we have

$$(\eta_{\mu\nu}(\partial_\rho\partial^\rho) - \partial_\mu\partial_\nu)A^\nu = 0.$$

Note that the operator $(\partial_{\mu\nu}(\partial_\rho\partial^\rho) - \partial_\mu\partial_\nu)$ is not invertible, since it annihilates any function of the form $\partial_\mu\lambda(x)$. So A^μ is *impossible to determine uniquely*; instead, it is determined *up to a choice of gauge*.

Definition: When A_μ can be reached from A_ν by a gauge transformation, we say they are in the same *gauge orbit*. Our configuration space of A_μ's is therefore *foliated by gauge orbits*.

Picking a gauge is the process of picking a point on each gauge orbit in a smooth manner. Our choice of gauge must intersect each gauge orbit exactly once:

Important examples of gauge choices are:

Definition: *Coulomb gauge* is specified by the condition $\nabla \cdot \mathbf{A} = 0$.

Theorem: Coulomb gauge is a valid choice of gauge.

Proof: If A_μ is our initial field, obeying $\nabla \cdot \mathbf{A} = 0$, suppose we want to gauge-transform to A'_μ. Generally, this will have $\nabla \cdot \mathbf{A}' = f(x)$ for some f. So $A'_\mu = A_\mu + \partial_\mu\lambda(x)$ implies that $\nabla^2\lambda = f$. This is Poisson's equation, so has solutions, so we can gauge-transform from A_μ to A'_μ. □

Definition: Lorentz gauge is specified by the condition $\partial_\mu A^\mu = 0$.

Theorem: Lorentz gauge is a valid choice of gauge.

Proof: Similar proof to Coulomb. □

Lorentz gauge has the advantage that it is manifestly Lorentz invariant. Coulomb gauge has the advantage that the condition $\nabla \cdot \mathbf{A} = 0$ makes it clear how another degree of freedom is absorbed, leaving two real degrees of freedom for the photon.

8.2 Quantisation in Lorentz gauge

The photon field is easiest to quantise in *Lorentz gauge*, $\partial_\mu A^\mu = 0$. How do we impose this gauge condition?

Theorem: Replacing the free Lagrangian $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

for some α, automatically imposes Lorentz gauge. The new equation of motion is:

$$\partial_\mu\partial^\mu A^\nu + \left(\frac{1}{\alpha} - 1\right)\partial^\nu\partial_\mu A^\mu = 0.$$

Proof: Lorentz gauge is imposed using the $1/\alpha$ equation of motion; that is, we must treat $1/\alpha$ as dynamical and consider the term we've added as a Lagrange multiplier. The equation of motion is found by standard methods. □

Definition: Confusingly, different choices of α are also referred to as different gauges. $\alpha = 1$ is called *Feynman gauge*, and $\alpha = 0$ (i.e. retaining only the second term $-\frac{1}{2}(\partial_\mu A^\mu)^2$) is called *Landau gauge*.

Theorem: The conjugate momenta are given by:

$$\pi^0 = -\frac{1}{\alpha}\partial_\mu A^\mu, \quad \pi^i = -\dot{A}^i + \partial^i A^0.$$

Proof: We have

$$\frac{\partial\mathcal{L}}{\partial_\sigma A_\rho} = -F^{\sigma\rho} - \frac{1}{\alpha}\eta^{\sigma\rho}\partial_\nu A^\nu.$$

Inserting $\sigma = 0$ and expanding $F^{\sigma\rho}$, we can read off the results. □

Using this Theorem, we can compute the classical Poisson bracket structure of the theory:

Theorem: $\{A_\mu(\mathbf{x}), A_\nu(\mathbf{y})\} = 0 = \{\pi_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})\}$, and

$$\{A_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})\} = \eta_{\mu\nu}\delta^3(\mathbf{x} - \mathbf{y}).$$

Proof: The field-theoretic Poisson bracket is:

$$\{f, g\} = \int d^3\mathbf{x}' \sum_i \left(\frac{\delta f}{\delta\phi_i(\mathbf{x}')}\frac{\delta g}{\delta\pi_i(\mathbf{x}')} - \frac{\delta f}{\delta\pi_i(\mathbf{x}')}\frac{\delta g}{\delta\phi_i(\mathbf{x}')}\right),$$

where the δ derivative is the variational derivative. The first two Poisson bracket identities are then obvious. The third is given by:

$$\begin{aligned}\{A_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})\} &= \int d^3\mathbf{x} \left(\frac{\delta A_\mu}{\delta A_\alpha}\frac{\delta\pi_\nu}{\delta\pi^\alpha} - \frac{\delta\pi_\nu}{\delta A_\alpha}\frac{\delta A_\mu}{\delta\pi^\alpha}\right) \\ &= \int d^3\mathbf{x}'\, \delta^\alpha{}_\mu \eta_{\alpha\nu}\delta^3(\mathbf{x} - \mathbf{x}')\delta^3(\mathbf{y} - \mathbf{x}') \\ &= \eta_{\mu\nu}\delta^3(\mathbf{x} - \mathbf{y}). \quad \square\end{aligned}$$

It is now a simple matter to write down the commutation relations of the quantised theory:

$$[A_\mu(\mathbf{x}), A_\nu(\mathbf{y})] = 0 = [\pi_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})],$$
$$[A_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})] = i\eta_{\mu\nu}\delta^3(\mathbf{x}-\mathbf{y}).$$

We now want to perform a mode expansion for $A_\mu(\mathbf{x})$ and $\pi_\mu(\mathbf{x})$. We make the usual mode expansion for $A_\mu(\mathbf{x})$:

$$A_\mu(\mathbf{x}) = \sum_{\lambda=0}^{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{p}|}}\left(\epsilon_\mu^\lambda(\mathbf{p})a_{\mathbf{p}}^\lambda e^{i\mathbf{p}\cdot\mathbf{x}} + \epsilon_\mu^{\lambda*}(\mathbf{p})a_{\mathbf{p}}^{\lambda\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}}\right).$$

Here, note the energy factor in the denominator has become $|\mathbf{p}|$ since photons are massless. The ϵ^λ vectors are called *polarisation vectors*.

WLOG, we may choose ϵ^0 to be timelike, and ϵ^i to be spacelike. We may also WLOG assume that the polarisation vectors obey the orthonormality relation:

$$\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'}.$$

We can choose the polarisation vectors to parallel classical electrodynamics. Choose ϵ^1 and ϵ^2 to be the transverse polarisations of the photon, i.e. $\epsilon^1\cdot p = \epsilon^2 \cdot p = 0$. Choose ϵ^3 to be the longitudinal polarisation, i.e. the polarisation in the direction of travel of the photon. Then if a photon has 4-momentum $p^\mu = |\mathbf{p}|(1,0,0,1)$, we may take WLOG the polarisation vectors to be:

$$\epsilon^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \epsilon^1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \epsilon^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \epsilon^3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

A useful result is the *completeness relation* for the polarisation vectors:

Theorem: We have:

$$\sum_{\lambda=0}^{3}\epsilon_\mu^\lambda(\mathbf{p})\epsilon_\nu^{\lambda*}(\mathbf{p}) = \eta_{\mu\nu}.$$

Proof: Use the basis above. Since the final result is basis independent, it holds for all possible polarisations. □

We have a similar relation, but only summing over the physical degrees of freedom:

Theorem: We have:

$$\sum_{\lambda=1}^{3}\epsilon_\mu^\lambda(\mathbf{p})\epsilon_\nu^{\lambda*}(\mathbf{p}) = -\eta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}.$$

Proof: PROOF STILL REQUIRED HERE. □

In the Heisenberg picture, the mode expansion becomes (as usual):

$$A_\mu(x) = \sum_{\lambda=0}^{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{p}|}}\left(\epsilon_\mu^\lambda(\mathbf{p})a_{\mathbf{p}}^\lambda e^{-ip\cdot x} + \epsilon_\mu^{\lambda*}(\mathbf{p})a_{\mathbf{p}}^{\lambda\dagger}e^{ip\cdot x}\right)$$

We can then compute: $\dot{A}_\mu(x) =$

$$-i\sum_{\lambda=0}^{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{\frac{|\mathbf{p}|}{2}}\left(\epsilon_\mu^\lambda(\mathbf{p})a_{\mathbf{p}}^\lambda e^{-ip\cdot x} - \epsilon_\mu^{\lambda*}(\mathbf{p})a_{\mathbf{p}}^{\lambda\dagger}e^{ip\cdot x}\right)$$

Recall $\pi^0 = -\frac{1}{\alpha}\dot{A}^0 - \frac{1}{\alpha}\nabla\cdot\mathbf{A}$, and $\pi^i = -\dot{A}^i + \partial^i A^0$. Since the A fields commute, they also commute with their spatial derivatives, so we can actually write the (equal-time) commutation relations above as:

$$[A_\mu(\mathbf{x},t), A_\nu(\mathbf{y},t)] = 0 = [\dot{A}_\mu(\mathbf{x},t), \dot{A}_\nu(\mathbf{y},t)],$$
$$[A_\mu(\mathbf{x},t), \dot{A}^0(\mathbf{y},t)] = -\alpha i\delta^0{}_\mu\delta^3(\mathbf{x}-\mathbf{y}),$$
$$[A_\mu(\mathbf{x},t), \dot{A}^i(\mathbf{y},t)] = -i\delta^i{}_\mu\delta^3(\mathbf{x}-\mathbf{y}).$$

Theorem: We have the following:

$$[a_{\mathbf{p}}^\lambda, a_{\mathbf{q}}^{\lambda'}] = 0 = [a_{\mathbf{p}}^{\lambda\dagger}, a_{\mathbf{q}}^{\lambda'\dagger}], [a_{\mathbf{p}}^0, a_{\mathbf{q}}^{i\dagger}] = 0,$$
$$[a_{\mathbf{p}}^0, a_{\mathbf{q}}^{0\dagger}] = -\alpha(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}),$$
$$[a_{\mathbf{p}}^i, a_{\mathbf{q}}^{j\dagger}] = \delta^{ij}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).$$

Proof: Long calculation; can verify quickly by substituting into above commutation relations and using the completeness relation for the polarisation vectors. □

8.3 The propagator

Theorem: The photon propagator is $\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+i\epsilon}\left(\eta_{\mu\nu} + (\alpha-1)\frac{p_\mu p_\nu}{p^2}\right)e^{-ip\cdot(x-y)}.$$

Proof: Set $x^0 > y^0$ and compute $A_\mu(x)A_\nu(y)$ sandwiched between $\langle 0|$ and $|0\rangle$. The calculation will reach the point:

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{-ip\cdot(x-y)}}{2|\mathbf{p}|}\left(-\alpha\epsilon_\mu^0(\mathbf{p})\epsilon_\nu^{0*}(\mathbf{p}) + \epsilon_\mu^i(\mathbf{p})\epsilon_\nu^{i*}(\mathbf{p})\right).$$

Use the completeness relation for the physical states $i = 1,2,3$ to evaluate $\epsilon_\mu^i(\mathbf{p})\epsilon_\nu^{i*}(\mathbf{p})$, and by subtracting the full completeness relation and the physical completeness relation, find that

$$\epsilon_\mu^0(\mathbf{p})\epsilon_\nu^{0*}(\mathbf{p}) = \frac{p_\mu p_\nu}{p^2}.$$

Introduce a contour integral over p^0 to get the remaining factors, as for the scalar propagator. □

8.4 The Gupta-Bleuler condition

Some of the commutation relations for the a's look unusual. Define $|p,\lambda\rangle = a_{\mathbf{p}}^{\lambda\dagger}|0\rangle$. Then

$$\langle p,\lambda=0|q,\lambda=0\rangle = \langle 0|a_{\mathbf{p}}^{0}a_{\mathbf{q}}^{0\dagger}|0\rangle = -\alpha(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).$$

So the state $\langle p,\lambda=0\rangle$ has *negative norm*. This is very bad.

To fix this, we need to use the constraint from the α equation; i.e. we need to impose the Lorentz gauge condition $\partial_\mu A^\mu = 0$. There are three ways of doing this:

1. Enforce this equation as an operator equation. Then $\pi^0 = -\frac{1}{\alpha}\partial_\mu A^\mu = 0$, and then our commutation relations break down.
2. Impose $\partial_\mu A^\mu|\psi\rangle = 0$ on all physical state $|\psi\rangle$. Again, this is too strong, since it's clear that $\partial_\mu A^\mu \sim a_{\mathbf{p}} - a_{\mathbf{p}}^\dagger$, so that $\partial_\mu A^\mu$ does not annihilate the vacuum. Thus not even the vacuum is a physical state in this regime!
3. Finally, writing:

$$A_\mu^+(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{p}|}}\sum_{\lambda=0}^{3}\left(\epsilon_\mu^\lambda(\mathbf{p})a_{\mathbf{p}}^\lambda e^{-ip\cdot x}\right),$$
$$A_\mu^-(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{p}|}}\sum_{\lambda=0}^{3}\left(\epsilon_\mu^{\lambda *}(\mathbf{p})a_{\mathbf{p}}^{\lambda\dagger} e^{ip\cdot x}\right),$$

we say that *physical states* $|\psi\rangle$ are those defined by $\partial_\mu A^{+\mu}(x)|\psi\rangle = 0$. Therefore, $\partial_\mu A^\mu$ has *vanishing matrix element between all physical states*: $\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0$.

Definition: The condition $\partial_\mu A^{+\mu}(x)|\psi\rangle = 0$ is called the *Gupta-Bleuler condition*.

Write $|\psi\rangle = |\psi_T\rangle|\phi\rangle$ for a generic state, where $|\psi_T\rangle$ contains all the *transverse* elements of the photon (i.e. those created by a^1 and a^2). By expanding $\partial_\mu A^{+\mu}$ completely, we find that the Gupta-Bleuler condition in the polarisation basis above is equivalent to:

$$(a_{\mathbf{k}}^3 - a_{\mathbf{k}}^0)|\phi\rangle = 0,$$

which means that physical states must contain combinations of longitudinal and timelike polarisations only.

Clearly this means that some states can have *zero norm*. This is also not very good; we thus treat these zero norm states as an equivalence class. Two states differing only in timelike/longitudinal pairs are treated as physically equivalent.

Indeed, no observables depend on pairs $|\phi\rangle$, e.g. the Hamiltonian of the theory is

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\left(\sum_{i=1}^{3} a_{\mathbf{p}}^{i\dagger}a_{\mathbf{p}}^{i} - a_{\mathbf{p}}^{0\dagger}a_{\mathbf{p}}^{0}\right).$$

Since $(a_{\mathbf{k}}^3 - a_{\mathbf{k}}^0)|\psi\rangle = 0$ on physical states, we see that $\langle\psi|a_{\mathbf{p}}^{3\dagger}a_{\mathbf{p}}^3 - a_{\mathbf{p}}^{0\dagger}a_{\mathbf{p}}^0|\psi\rangle = 0$. So the timelike and longitudinal pieces *cancel* leaving only the transverse contribution.

8.5 Coupling to fermions

Theorem: Any operator coupled to a photon must be a conserved current.

Proof: Interaction terms in the Lagrangian coupling to photon appear as $-j^\mu A_\mu$. These give rise to the classical equation of motion $\partial_\mu F^{\mu\nu} = j^\nu$ when added to the Lagrangian, and hence taking the derivative ∂_ν of both sides, we see $\partial_\nu j^\nu = 0$. □

Thus if we want to add fermions to our theory, we need to add fermionic conserved currents. But we have loads of these! An example is $\overline{\psi}\gamma^\mu\psi$.

Definition: The Lagrangian of *quantum electrodynamics* (QED) is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}\gamma^\mu A_\mu\psi,$$

where e is some coupling constant.

It's crucial that this is gauge invariant. We can see that it is by defining:

Definition: The *covariant derivative* is defined by

$$D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi.$$

In terms of the covariant derivative, the QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(iD\!\!\!/ - m)\psi.$$

Theorem: Under a gauge transformation $\psi \mapsto e^{-ie\lambda(x)}\psi$ of the spinor field, the expression $D_\mu\psi$ transforms to $e^{-ie\lambda(x)}D_\mu\psi$.

Proof: We have

$$D_\mu\psi \mapsto \partial_\mu(e^{-ie\lambda(x)}\psi) + ie(A_\mu + \partial_\mu\lambda(x))e^{-ie\lambda(x)}\psi =$$
$$e^{-ie\lambda(x)}\partial_\mu\psi - ie\psi(\partial_\mu\lambda(x))e^{-ie\lambda(x)}\psi + ie(A_\mu + \partial_\mu\lambda(x))e^{-ie\lambda(x)}\psi$$
$$= e^{-ie\lambda(x)}D_\mu\psi,$$

as required. □

From this result, it's easy to see that the QED Lagrangian is indeed gauge invariant.

The coupling constant e has an important interpretation. Since the equations of motion here are:

$$\partial_\mu F^{\mu\nu} = e\overline{\psi}\gamma^\nu\psi,$$

it follows that we have a conserved charge:

$$Q = -e\int d^3\mathbf{x}\overline{\psi}\gamma^0\psi = -e\int \frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{s=1}^{2}\left(b_\mathbf{p}^{s\,\dagger}b_\mathbf{p}^s - c_\mathbf{p}^{s\,\dagger}c_\mathbf{p}^s\right).$$

Hence $Q = -eN$ is conserved, where N is the different between the number of particles and anti-particles. It appears that e should have the interpretation of the *electric charge* of the fermion. Particles have the opposite charge of their anti-particles.

When we calculate cross-sections and decay rates later on, it will be useful to define:

Definition: The *fine-structure constant* is

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}.$$

It's easy to write down the Feynman rules of QED:

1. The photon propagator is given by:

$$-\frac{i}{p^2}\left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right)$$

   In *Feynman gauge* ($\alpha = 1$), this is just $-i\eta_{\mu\nu}/p^2$.
2. For an incoming or outgoing photon, add a polarisation vector $\epsilon^\mu_{\text{in}}/\epsilon^\mu_{\text{out}}$.
3. The only interaction vertex, between two fermions and a photon, has value $-ie\gamma^\mu$.

8.6 Coupling to scalars

For a complex scalar ϕ, note that if we define the covariant derivative by

$$D_\mu\phi = \partial_\mu\phi - ieqA_\mu\phi,$$

where q is the charge of ϕ in units of e, we have under a gauge transformation $\phi(x) \mapsto e^{ieq\lambda(x)}\phi(x)$ that $D_\mu\phi \mapsto e^{ieq\lambda(x)}D_\mu\phi$. Thus

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^\dagger$$

is a gauge invariant Lagrangian.

Definition: The above Lagrangian is the Lagrangian of *scalar electrodynamics*.

Writing out the Lagrangian in full, we have the interaction terms:

$$\mathcal{L}_{\text{int}} = ieq(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi)^\dagger\phi)A_\mu + e^2q^2A_\mu A^\mu\phi^\dagger\phi.$$

This immediately gives the vertex rules:

1. There is a vertex where two photons and two scalar particles meet. The factor is $-ie^2q^2$.
2. There is a vertex where two scalars and a photon meet. This has a derivative coupling. Since

$$\partial^\mu\phi = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left((-ip^\mu)b_\mathbf{p}e^{-ip\cdot x} + (ip^\mu)c_\mathbf{p}{}^\dagger e^{ip\cdot x}\right),$$

   we get a factor of eq and:
   - $-ip^\mu$ for an incoming scalar of momentum p^μ;
   - ip^μ for an outgoing anti-scalar of momentum p^μ;
   - ip^μ for an incoming anti-scalar of momentum p^μ (from $(\partial^\mu\phi)^\dagger$ term);
   - $-ip^\mu$ for an outgoing scalar of momentum p^μ (again, from $(\partial^\mu\phi)^\dagger$ term).

8.7 Example

Example: Consider electron to muon scattering: $e^-(p,s)e^+(q,r) \to \mu^-(p',s')\mu^+(q',r')$. There is only one diagram for this process, since electrons and muons don't couple. The Feynman rules give:

$$iA_{fi} = -\frac{i(-ie)^2}{s}[\bar{u}_m^{s'}(\mathbf{p}')\gamma_\mu v_m^{r'}(\mathbf{q}')][\bar{v}_e^r(\mathbf{q})\gamma^\mu u_e^s(\mathbf{p})].$$

Neglecting masses of the fermions, and calculating the spin-sum average we have:

$$\overline{|A|^2} = \frac{2e^4}{s^2}(u^2 + t^2).$$

(It's possible to use fermion loop techniques here.)

The differential cross section is therefore:

$$\frac{d\sigma}{dt} = \frac{e^4(u^2 + t^2)}{8\pi s^4}.$$

Since we are working in the massless limit, $u = -s - t$, and thus

$$\frac{d\sigma}{dt} = \frac{e^4(2t^2 + 2st + s^2)}{8\pi s^4}.$$

Also $s = (p+q)^2 = 2p\cdot q = 4|\mathbf{p}|^2$ (assuming $\mathbf{p} = -\mathbf{q}$, i.e. in the COM frame) and

$$t = (p - p')^2 = -2p\cdot p' = 2|\mathbf{p}||\mathbf{p}'|(\cos(\theta) - 1).$$

Using conservation of energy: $2\sqrt{|\mathbf{p}|^2} = 2\sqrt{|\mathbf{p}'|^2} \Rightarrow |\mathbf{p}| = |\mathbf{p}'|$, so all momentum have the same magnitude in this problem. Thus $t = \frac{1}{2}s(\cos(\theta) - 1)$. When $\theta = 0$, get $t = 0$. When $\theta = \pi$, get $t = -s$. So integrate in the range $[-s, 0]$.

Doing so gives the final answer: $\sigma = 4\pi\alpha^2/3s$ (on replacing e^2 by α appropriately).

Any corrections to the amplitude would be at loop order with 4 vertices, i.e. $O(e^4) = O(\alpha^2)$. Hence the amplitude would become $A = \alpha A^0 + \alpha^2 A^1 + \cdots$, and thus the cross section would be corrected at order $O(\alpha^3)$. This is very small!