Part IB: Complex Analysis
Bonus Questions

Please send all comments and corrections to jmm232@cam.ac.uk.

1. Define $g : \mathbb{C} \to \mathbb{C}$ by $g(0) = 0$ and $g(z) = e^{-1/z^4}$ for $z \neq 0$. Show that g satisfies the Cauchy-Riemann equations everywhere, but is neither continuous nor differentiable at 0.

2. Define the differential operators:
$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$
   (i) Prove that a C^1 function f is holomorphic if and only if $\partial f/\partial \bar{z} = 0$.
   (ii) Show that:
$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z},$$
   where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the usual Laplacian in $\mathbb{R}^2$.
   (iii) Let $f : U \to V$ be holomorphic and let $g : V \to \mathbb{C}$ be harmonic. Show that the composition $g \circ f$ is harmonic.

3. Let $U \subseteq \mathbb{C}$ be open and let $f = u + iv : U \to \mathbb{C}$. Suppose that u and v are C^1 on U as real functions of the real variables x, y, where $x + iy \in U$. Let $w \in U$ and suppose that the map f is angle-preserving at w in the following sense: for any two C^1 curves $\gamma_1, \gamma_2 : (-1, 1) \to U$ with $\gamma_j(0) = w$ and $\gamma_j'(0) \neq 0$ for $j = 1, 2$, the curves $\alpha_j = f \circ \gamma_j = u \circ \gamma_j + iv \circ \gamma_j$ satisfy $\alpha_j'(0) \neq 0$ and
$$\arg \frac{\alpha_1'(0)}{\gamma_1'(0)} = \arg \frac{\alpha_2'(0)}{\gamma_2'(0)}.$$
   Show that f is complex differentiable at w with $f'(w) \neq 0$. [You may find it useful to employ the operator $\partial/\partial\bar{z}$ in Q2.]

4. Use the (real) inverse function theorem (from the Analysis & Topology course) to prove the following holomorphic inverse function theorem: if $U \subseteq \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$ for some $z_0 \in U$, then there is an open neighbourhood V of z_0 and an open neighbourhood W of $f(z_0)$ such that $f|_V : V \to W$ is a bijection with holomorphic inverse. [Use the fact that holomorphic functions are C^1, i.e. have C^1 real and imaginary parts; this is proved - in fact that holomorphic functions are infinitely differentiable - in the course.]