Introduction to symmetry in field theories

Abstract

The Part III Symmetries, Fields and Particles course is about symmetry as applied to our modern theories of particle physics. Whilst most of the course will focus on mathematics, in this introductory handout we will attempt to motivate *why* we need the theory that follows.

We begin by discussing *field theory*, the language in which all modern particle physics is written. Field theories possess two types of symmetries: *external* and *internal* symmetries (and in turn, internal symmetries can be classified as either *global* or *gauge* symmetries). The fields in a field theory transform under *representations* of the external and internal symmetry groups; this will be an important theme throughout the course.

In the second part of the handout, we describe in detail the symmetry properties of a specific field theory, namely the *Standard Model of particle physics*.

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1 Symmetry in field theories

All modern theories of particle physics are *field theories*, where the basic dynamical variables of interest are *fields*, defined as follows:

> **Definition 1.1:** Let M be a model of spacetime. A *field* is a map $\phi : M \to N$ into some set N, called the *target space* of the field.

This definition is a little bit silly, because it simply says, in the strict mathematical sense, a field is the same thing as a function. The important thing is the *interpretation* of the domain and codomain of this function; there is a contrast here with *particle theories*, which are described by trajectories $x : \mathbb{R} \to M$ taking values *in* spacetime.

Let's try to be a little bit more specific by restricting the types of mathematical objects that the domain and codomain can be (though be aware that these restrictions are not always appropriate):

- Typically, we take spacetime M to be some *Lorentzian manifold*.[1] A *manifold* is a space which looks locally like $\mathbb{R}^n$, and where we can develop a theory of real multivariable calculus; manifolds are defined properly in a separate handout. The word *Lorentzian* refers to the fact that there is a 'local symmetric bilinear form' (the *metric*) on this manifold, and that this metric has signature $(1, -1, -1, ..., -1)$. More details can be found in the Part III General Relativity course.

  In Standard Model physics (where we ignore the possibility of curved spacetime) we take $M = (\mathbb{R}^4, \eta)$, i.e. $\mathbb{R}^4$ equipped with the Minkowski metric $\eta = \mathrm{diag}(1, -1, -1, -1)$.

- The target space is normally taken to be some *vector bundle*; we will not develop this theory here, but more details can be found in a differential geometry text.

Throughout this course, we will mainly focus on the case where M is Minkowski spacetime and N is a finite-dimensional real or complex vector space.

With the basic definition of a field out of the way, we can give a broad definition of a symmetry of a field theory:

> **Definition 1.2:** A *symmetry* of a field theory is a transformation of the fields which leaves all predictions of the theory (e.g. cross-sections and decay rates) invariant.

[1] Whilst this might seem a very minimal assumption, this rules out lattice theories, for example.

We now discuss two broad classes of symmetries of field theories.

Internal symmetries

One type of symmetry in a field theory is an *internal symmetry*, which only involves a transformation of the target space:

Definition 1.3: Suppose that $\phi : M \to N$ is a field in a field theory with spacetime M and target space N. An *internal symmetry* is is a transformation $T : N \to N$ of the target space such that $\phi \mapsto T \circ \phi$ is a symmetry of the field theory.

A good example of an internal symmetry in a field theory is the *colour symmetry* of quarks:

Example 1.4: A *quark* is a fundamental particle, from which *hadrons* (such as protons and neutrons) are made. Quarks arise as excitations of an underlying *quark field* which fills spacetime; let us write this field as $q(x)$ (in fact, there are different *flavours* of quark - so there are really fields for the up quark, down quark, etc, but we won't worry about that here). The quark field can be viewed as a map from Minkowski spacetime into a target space called the *colour space*, $q : M \to \mathbb{C}^3$ (actually, the space on the right hand side is a tensor product of $\mathbb{C}^3$ with other complicated ingredients related to how quarks transform under weak isospin symmetry and Poincaré symmetries, but we'll ignore those pieces here). This means that we can view the quark field as being a vector-valued function:

$$q(x) = \begin{pmatrix} q_r(x) \\ q_b(x) \\ q_g(x) \end{pmatrix}.$$

The components of the colour space are affectionately known as *red*, *green* and *blue*. If an excitation of the quark field is completely aligned with the red direction, that excitation is referred to as a *red quark*, for example.

The theory built out of the quark field(s) (when all flavours are included) is called *quantum chromodynamics*. It is constructed so that the transformation:

$$q(x) \mapsto Uq(x),$$

for any matrix $U \in SU(3)$, is a symmetry of the theory - this symmetry is an example of an internal symmetry called *colour symmetry*.

In order to insert interactions into the theory, we follow a procedure known as '*gauging* the symmetry'. The key step is to demand that our 'colour rotation' U is not just a *global* rotation of the colour space, but can instead vary smoothly from point to point (so it is a *local* symmetry). That is, the quark fields instead transform as:

$$q(x) \mapsto U(x)q(x),$$

where $U : M \to SU(3)$ is now any smooth function from spacetime into $SU(3)$. Remarkably, this is still a symmetry of the theory, provided that we introduce interactions between the quarks, mediated by a *massless gauge boson* called the *gluon*. These interactions can be visualised using *Feynman diagrams*, for example, the diagram for quark-antiquark fusion into a gluon is given in Figure 1.

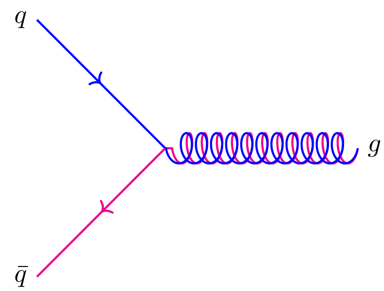

Figure 1: *Quark-antiquark fusion to produce a gluon, keeping track of the conserved colour charge.*

Applying Noether's theorem to colour symmetry, we find that we get a conserved *colour charge*. Hence, colour must be conserved in all interactions. For example, in Figure 1, we have shown a blue quark fusing with an antigreen (magenta) antiquark, to produce a blue-antigreen gluon (gluons must carry *two* colours to ensure conservation of colour charge).

The above colour example shows us that we can make a further useful distinction between *global* internal symmetries and *local* (or *gauge*) internal symmetries:[2]

Definition 1.5: If an internal symmetry is the same at all spacetime points, it is called a *global symmetry*. If an internal symmetry varies smoothly between spacetime points, it is called a *local* or *gauge* symmetry.

External symmetries

Another natural way of generating symmetries of a field theory is to consider transformations of spacetime itself:

Definition 1.6: Suppose that $\phi : M \to N$ is a field in a field theory with spacetime M and target space N. An *external symmetry* is a transformation $T : M \to M$ of spacetime such that:

$$\phi \quad \mapsto \quad R(T) \circ \phi \circ T^{-1}$$

for some *induced* transformation $R(T) : N \to N$ of the target space N.

The transformation law above, namely $\phi \mapsto R(T) \circ \phi \circ T^{-1}$, initially looks a little strange. There are two things that might bother you:

- Why do we precompose the field with T^{-1}? This ensures that the field takes the same value at the same point before and after the transformation (up to the induced transformation of the target space); that is, the transformation is a *passive* change of coordinates on spacetime. Explicitly, the field initially takes the value $\phi(x)$ at the point $x \in M$ before the transformation. After the transformation, the point $x \in M$ is relabelled as $T(x) \in M$. The field adjusts accordingly; the value of the transformed field $\phi \circ T^{-1}$ at $T(x) \in M$ is still $\phi(T^{-1}(T(x)) = \phi(x)$.

[2]The particularly alert reader will have noticed that gauge symmetries are *not* actually internal symmetries in the way we have defined above. Indeed, we said that an internal symmetry was a transformation purely of the target space N, i.e. a transformation $T : N \to N$, so it cannot in fact know anything about spacetime! This is remedied by replacing the field ϕ by the field $\tilde{\phi} : M \to M \times N$, which satisfies $\tilde{\phi}(x) = (x, \phi(x))$. The 'local' transformation $\phi(x) \mapsto U(x)\phi(x)$ can then be replaced by an honest, internal symmetry defined by the map $T(x, \phi(x)) = (x, U(x)\phi(x))$. We'll completely ignore this subtlety for now.

In fact, even the replacement of ϕ by $\tilde{\phi} : M \to M \times N$ is, in general, a white lie; a proper discussion requires the replacement of ϕ by a new field $\tilde{\phi} : M \to P$, which maps into a *fibre bundle* P that looks *locally* like $M \times N$. The technicalities are discussed in David Skinner's Part III Advanced Quantum Field Theory notes, in the section on classical Yang-Mills theories.

- How can it be the case that an external symmetry, i.e. a transformation of spacetime, somehow induces a transformation on the target space as well? This is most naturally seen when we consider *vector*-valued fields. For example, consider the vector field $\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$. We can view $\boldsymbol{\phi}$ as attaching a vector to each point of $\mathbb{R}^3$. Under a rotation of the 'spacetime' $\mathbb{R}^3$, say $\mathbf{v} \mapsto A\mathbf{v}$, it is natural to ask that the vectors attached to each of the points of $\mathbb{R}^3$ rotate in the same way. Thus we should ask that $\boldsymbol{\phi}$ transforms as $\boldsymbol{\phi}(\cdot) \mapsto A\boldsymbol{\phi}(A^{-1}\cdot)$. In this way, a 'natural' induced transformation appears in the target space.

If we have a collection of external symmetries which we implement in our field theory, the associated induced transformations cannot be completely arbitrary. They must respect *composition* of external symmetries.

Proposition 1.7: Let $T_1, T_2 : M \to M$ be two external symmetries for a field theory with fields $\phi : M \to N$. Then the induced transformations $R(T_1), R(T_2) : N \to N$ of the target space must obey:

$$R(T_1) \circ R(T_2) = R(T_1 \circ T_2).$$

Proof: Simply consider applying the external symmetry T_2 to the theory followed by the external symmetry T_1, versus applying the composed external symmetry $T_1 \circ T_2$ all in one go. □

When N is a vector space and the induced transformations are invertible linear maps, the above property makes the map:

$$R : \{\text{external symmetries}\} \to \{\text{invertible linear maps } N \to N\}$$

a *representation*;[3] we will study representations in great detail later in the course.

Specialising to particle physics, we are most interested in the case when M is Minkowski spacetime and N is a real or complex finite-dimensional vector space. In this case, the natural transformations of spacetime are the *Poincaré transformations* $T_{(\Lambda,a)} : M \to M$, given by

$$T_{(\Lambda,a)}(x) = \Lambda x + a,$$

where $a \in \mathbb{R}^4$ is a four-translation and Λ is a Lorentz transformation matrix. Poincaré transformations are precisely the transformations of Minkowski spacetime which preserve the Minkowski metric; we shall remind the reader of this fact later in the course. Fields can be classified by which representation their induced transformations on target space follow:

Definition 1.8: Let $\phi : M \to N$ be a field on Minkowski spacetime M, let N be a real or complex finite-dimensional vector space, and suppose that under a Poincaré transformation $T_{(\Lambda,a)}$ we have:

$$\phi \quad \mapsto \quad R(T_{(\Lambda,a)})\phi \circ T^{-1}_{(\Lambda,a)},$$

for some matrix $R(T_{(\Lambda,a)})$. We define the following types of fields:

- A *scalar field* has $N = \mathbb{R}$ or $\mathbb{C}$ and $R(T_{(\Lambda,a)}) = 1$. We say that the field transforms in the *trivial representation* under Poincaré transformations.
- A *vector field* has $N = \mathbb{R}^4$ or $\mathbb{C}^4$ and $R(T_{(\Lambda,a)}) = \Lambda$. We say that the field transforms in the *vector representation* under Poincaré transfromations.

More general types of fields are possible; in order to discover these, we need to *classify* the possible representations of the collection of Poincaré transformations. This may be covered later in the course; it is certainly covered in the first few chapters of Weinberg's excellent text *The Quantum Theory of Fields, Volume I*, otherwise.

[3] For those who already have exposure to group theory, observe that 'representation' is just a fancy word for 'group homomorphism into the group of automorphisms of a vector space'.

2 Symmetry in the Standard Model of particle physics

The Standard Model of particle physics can be described in terms of its symmetries as a Poincaré-invariant[4] $SU(3)\times SU(2)\times U(1)$ gauge theory with certain matter fields (to be described below), together with a scalar boson - called the *Higgs boson* - which induces a symmetry-breaking mechanism reducing the $SU(2)\times U(1)$ symmetry to a $U(1)$ symmetry.

To understand these words, we need to break down each of the pieces carefully:

- The fact that the theory is Poincaré invariant implies that the fields in the theory must be scalars, vectors, or *spinors*[5] (which we shall meet later in the course, and which you shall also meet in Part III Quantum Field Theory), all of which transform in different representations under Poincaré transformations. It is possible to show that when scalar fields are excited they give rise to spin-0 particles, when vector fields are excited they give rise to spin-1 particles, and when spinor fields are excited they give rise to spin-$\frac{1}{2}$ particles.
- The set $SU(3)\times SU(2)\times U(1)$ is called the *gauge group*[6] of the Standard Model. As in the colour symmetry example above, each factor of the gauge group contributes a *massless gauge boson* to the theory, which mediates some interaction between the matter fields (described next). In particular:
  - The factor $SU(3)$ is called the *colour symmetry group* of the Standard Model. As we have already mentioned, the massless gauge boson that arises from this gauge group is the *gluon*, g. There are 8 different types of gluon, each carrying different combinations of the *colour* charge. The interaction mediated by the gluons is called the *strong interaction*.
  - The factors $SU(2)\times U(1)$ are collectively called the *electroweak symmetry group* of the Standard Model. It comprises $SU(2)$, called the *weak isospin group*, and $U(1)$, called the *weak hypercharge group*. Both of these have massless gauge bosons associated with them, but they *do not appear in Nature!* The reason for this is that $SU(2)\times U(1)$ is *broken* in Nature by an asymmetry caused by the *Higgs boson*, h, so that only a $U(1)$ gauge symmetry remains.

    The remaining factor $U(1)$ contributes the familiar *photon*, γ, to the theory. The interaction mediated by the photons is, of course, *electromagnetism*.

    The broken part of the symmetry still contributes some bosons to the theory, but they are now *massive* bosons, called the W^+, W^- and Z-bosons. The interaction mediated by the W^+, W^- and Z-bosons is called the weak interaction.
- The *matter content* of the Standard Model consists of fields which transform under Poincaré transformations as spinor fields (so they are spin-$\frac{1}{2}$ fields, i.e. fermion fields). They are distinguished by their transformation properties under $SU(3)\times SU(2)\times U(1)$. We won't go into detail here, but note:
  - *Coloured particles* transform non-trivially under $SU(3)$, whilst *colourless particles* transform trivially under $SU(3)$. *Left-handed particles* transform non-trivially under $SU(2)$, whilst *right-handed particles* transform trivially under $SU(2)$. *Electrically-charged particles* transform non-trivially under $U(1)$, whilst *electrically-neutral particles* transform trivially under $U(1)$.
  - The Standard Model contains three 'copies' of each matter particle, which all transform in exactly the same way under $SU(3)\times SU(2)\times U(1)$. The only property that distinguishes these copies is the *mass* of each of the particles. These three copies are called *generations*, and no-one knows why they exist.
  - All of the following particles also come with respective *anti-particles*. Anti-particles are denoted by placing a bar on top of the symbol for the particle, e.g. an anti-electron is denoted $\bar{e}$.

[4] When we say that a theory is *Poincaré invariant*, we typically mean that the theory is invariant under only Poincaré transformations which are *connected to the identity* - we shall see that this means transformations such as parity and time-reversal are out. This is the sense in which we use the phrase Poincaré invariant when describing the Standard Model.

[5] Technically, we can also have tensor fields, such as the hypothetical spin-2 *graviton field* $g_{\mu\nu}$, however these theories become increasingly sick as the number of tensor indices increases. In the Standard Model, there are only scalar, spinor and vector fields.

[6] We will define a *group* later in the course, for those who have not had prior exposure to group theory.

- *Quarks* are coloured, electrically-charged particles. They can be left-handed or right-handed, both feature in the Standard Model. There are two possible charges for quarks, either $+2/3$ for the up, u, charm, c, and top, t, quarks (note the three generations - only mass distinguishes between up, charm and top quarks), and $-1/3$ for the down, d, strange, s, and bottom, b, quarks (again, note the three generations).
- *Charged leptons* are colourless, electrically-charged particles. They can be left-handed or right-handed. They each carry a charge -1, and in order of increasing mass they are electrons, e, muons, μ, and taus, τ.
- *Neutrinos* are colourless, electrically-neutral particles. They are only known to exist in their left-handed variety (right-handed neutrinos would not interact in the Standard Model). They are the electron neutrino, ν_e, the muon neutrino, ν_μ, and the tau neutrino, ν_τ.

This completes our description of the Standard Model of particle physics, viewed in terms of its rich symmetry structure; a summary of the particle content is given in Figure 2. During the course of Part III, you will deepen your understanding of this structure; the Part III Symmetries, Fields and Particles course should provide a key step in that direction.

SUMMARY OF PARTICLES IN THE STANDARD MODEL

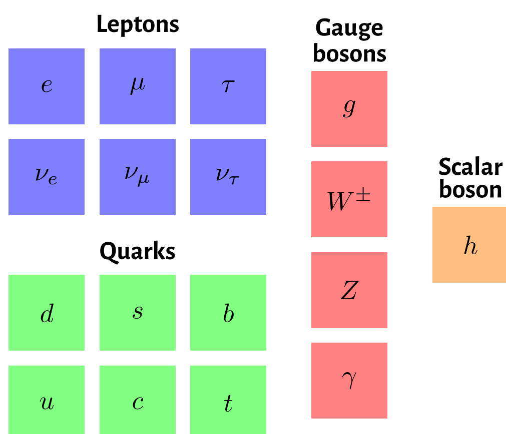

Figure 2: *The particles in the Standard Model of particle physics. See the text for more details.*