Manifolds

Abstract

In the first handout, we mentioned that the types of groups in which we are typically interested in physics are *Lie groups*, which are groups which can be 'parametrised in terms of some *smooth* coordinates' (the word 'smooth' implies some ability to *calculus* on the group). In this handout, we make the notion of spaces which can be 'parametrised by smooth coordinates' precise; such spaces are called *manifolds*. This prepares the way for a full discussion of Lie groups later in the course.

The basic idea is that, locally, a manifold looks like an open subset of $\mathbb{R}^n$; in particular, this allows us to port all definitions from real multivariable calculus to the manifold, including notions such as *smoothness*. We begin by defining manifolds and giving various examples. We then follow the standard sequence of steps when we define a new mathematical structure: we begin by discussing how to obtain new instances of manifolds from existing instances, before discussing natural maps between manifolds. We close by discussing some useful properties that manifolds inherit from topology, namely connectedness, path-connectedness, simply-connectedness and compactness.

1 Manifolds: definitions and examples

As described above, we would like a manifold to be a space which looks locally like an open subset of $\mathbb{R}^n$. With this in mind, we posit the following initial definition:

Definition 1.1: A *topological n-manifold X* is a topological space satisfying the following axioms:

(M1) The topology on X is *Hausdorff*: given any two points $p, q \in X$, there exists open neighbourhoods $U \ni p, V \ni q$ such that $U \cap V \neq \emptyset$ (i.e. 'points can be separated by open sets').

(M2) The topology on X is *second-countable*: the topology is generated by a countable basis of open sets.

(M3) The topology on X is *locally n-Euclidean*: for all points $p \in X$, there exists an open neighbourhood $U \ni p$ on which there is some homeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$. The pair (U, ϕ) is called a *chart*, and the map ϕ is called a set of *local coordinates* on the domain U. The inverse $\phi^{-1} : \phi(U) \to U$ is called a *parametrisation* of the domain U.

The integer n is called the *dimension* of the topological n-manifold.[1]

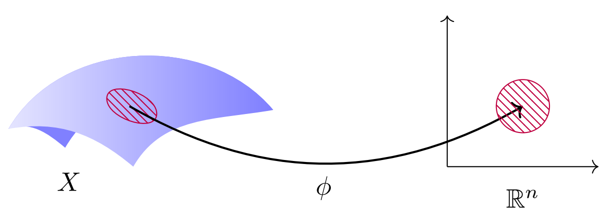

Figure 1: *A topological n-manifold is a topological space which looks locally like an open subset of $\mathbb{R}^n$.*

The first two axioms (M1) and (M2) are purely technical, and ensure the statements of some theorems are not too cluttered. The important axiom realising our intuition was (M3).

[1] It can be shown using methods from algebraic topology that if a topological space is both a topological n-manifold and a topological m-manifold, then $n = m$; therefore, the dimension of a topological n-manifold is well-defined.

Topological n-manifolds are not yet spaces on which we can do calculus. Suppose, for example, that we wish to define smoothness of a real-valued function $f : X \to \mathbb{R}$ on the topological n-manifold X at a point p. Reasonably, we might declare that f is smooth at the point $p \in X$ if there is some chart (U, ϕ) whose domain contains p such that the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth as a real, multivariable function (i.e. all its partial derivatives exist to all orders). We think of $f \circ \phi^{-1}$ as the 'local coordinate expression of the function f'.

However, this definition is a little unsatisfying. Suppose that we change coordinates about p, so that we now work with another chart (V, ψ) whose domain contains p. Since our only assumption on ϕ and ψ is that they are *homeomorphisms*, the most we can say about the 'change of coordinates function' $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is that it is a *continuous function*. In particular it does not follow that $f \circ \psi^{-1} = f \circ \phi^{-1} \circ \phi \circ \psi^{-1}$ is necessarily smooth as a real, multivariable function. Changing coordinates can change the smoothness of a function!

To remedy this, we restrict to a collection of charts where *changes of coordinates* are smooth. In particular, we choose some subset of charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$, whose domains U_α cover X, such that for any charts (U_α, ϕ_α), (U_β, ϕ_β), the transition function $\phi_\alpha \circ \phi_\beta^{-1}$ is a smooth function in the sense of real, multivariable calculus. This fixes the problem, since if $f \circ \phi_\alpha^{-1}$ is smooth, then $f \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1}$ is smooth by the chain rule.

Baking this into a definition, we have the following:

Definition 1.2: Let X be a topological n-manifold. A *smooth atlas* on X is a collection of charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$, whose domains U_α cover X, such that for any charts (U_α, ϕ_α), (U_β, ϕ_β) the *transition function*:

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a smooth function in the sense of real, multivariable calculus.

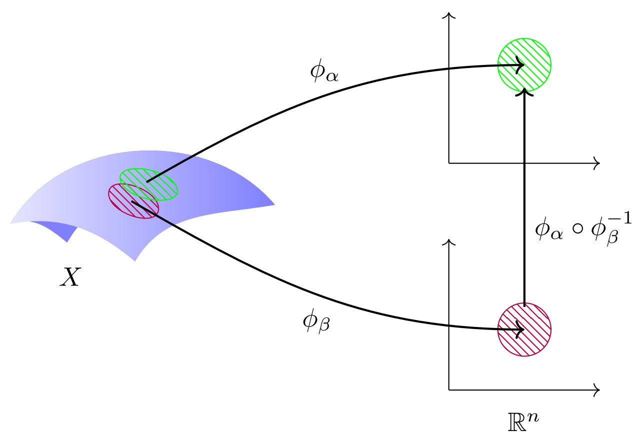

Figure 2: *A smooth atlas on a topological n-manifold is a collection of charts for which all transition functions, i.e. changes of coordinates, are smooth (in the sense of real, multivariable calculus).*

We can now define smooth functions on a topological n-manifold *with respect to a smooth atlas*, avoiding the problem we described above.

Definition 1.3: Let X be a topological n-manifold, and let $\mathcal{A}$ be a smooth atlas on X. A function $f : X \to \mathbb{R}$ is *smooth* with respect to $\mathcal{A}$ if for some chart $(U, \phi) \in \mathcal{A}$ we have that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is a smooth function in the sense of real multivariable calculus.

We are now in a good position; by choosing a special subset of charts on a topological n-manifold, we can reasonably begin to talk about smooth functions. However, it is quite disappointing that our choice is arbitrary - would another choice have led to the same set of smooth functions on the topological n-manifold?

To remove the arbitrariness, we introduce a natural notion of *equivalence* of smooth atlases:

Definition 1.4: Let X be a topological n-manifold and let $\mathcal{A}, \mathcal{B}$ be smooth atlases on X. We say that $\mathcal{A}, \mathcal{B}$ are *smoothly equivalent* if for all functions $f : X \to \mathbb{R}$, f is smooth with respect to $\mathcal{A}$ if and only if f is smooth with respect to $\mathcal{B}$. That is, $\mathcal{A}, \mathcal{B}$ give rise to the same smooth functions on X.

Naturally, smooth equivalence of smooth atlases is an equivalence relation:

Proposition 1.5: Smooth equivalence of smooth atlases is an equivalence relation on the set of smooth atlases. We call an equivalence class of smooth atlases a *smooth structure*.

Proof: Easy exercise. $\square$

Now, instead of saying that a function is smooth with respect to some arbitrary choice of smooth atlas, we can reliably say that a function is smooth with respect to some *smooth structure*. This (finally) leads us to make the definition of a *smooth n-manifold*:

Definition 1.6: A *smooth n-manifold* (henceforth abbreviated to n-manifold, or just manifold when n is clear) is a topological n-manifold equipped with a smooth structure. The integer n is called the *dimension* of the manifold.

As an interesting aside, it turns out that topological $1, 2$ and 3-manifolds can be given a unique smooth structure. Topological n-manifolds with $n \geq 4$ can be given multiple, inequivalent smooth structures.

Examples of manifolds

The most basic example of a manifold is $\mathbb{R}^n$ itself.

Example 1.7: The standard topology on $\mathbb{R}^n$ is Hausdorff (since it is induced by a metric) and second-countable (a basis for the topology is provided by balls with rational centres and rational radii). Given any point $p \in \mathbb{R}^n$, we have that $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is a chart whose domain contains p. Therefore $\mathbb{R}^n$ is a topological n-manifold.

The chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is a *global* chart since it covers all of $\mathbb{R}^n$. In particular, $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$ is trivially a smooth atlas on $\mathbb{R}^n$, since the only transition function is the trivial transition function. Hence $\mathbb{R}^n$ can be made into an n-manifold, with a representative of its smooth structure given by $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$.

The above example allows us to immediately write down many related examples:

Example 1.8:

- $\mathbb{C}^n$ is a $2n$-manifold. To see this, we simply note $\mathbb{C}^n = \mathbb{R}^{2n}$, by writing each component of a vector in $\mathbb{C}^n$ in terms of its real and imaginary parts.
- The space of $n \times n$ matrices over $\mathbb{R}$, $\mathrm{Mat}_n(\mathbb{R})$, is an n^2-manifold. Again, this is because we can identify $\mathrm{Mat}_n(\mathbb{R}) = \mathbb{R}^{n^2}$, simply by stacking all entries of a matrix in a column vector in some way.
- The space of $n \times n$ matrices over $\mathbb{C}$, $\mathrm{Mat}_n(\mathbb{C})$, is a $2n^2$-manifold. This follows from the chain of equalities $\mathrm{Mat}_n(\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$, using both of the identifications in the previous two bullet points.

A less trivial example of a manifold is the circle, S^1.

Example 1.9: Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. We can endow S^1 with the subspace topology, inherited from its ambient space $\mathbb{C} = \mathbb{R}^2$; in particular, its topology is therefore Hausdorff and second-countable. We define two charts on S^1:

- $\phi : S^1 \backslash \{1\} \to (0, 2\pi)$ given by $\phi(e^{i\theta}) = \theta$ for $\theta \in (0, 2\pi)$.
- $\psi : S^1 \backslash \{-1\} \to (-\pi, \pi)$ given by $\psi(e^{i\theta}) = \theta$ for $\theta \in (-\pi, \pi)$.

Figure 3: *Two charts covering S^1. On their overlap, their transition functions must be smooth.*

The existence of these charts shows that S^1 is locally 1-Euclidean, hence S^1 is a topological 1-manifold. Furthermore, $\{(S^1\backslash\{1\}, \phi), (S^1\backslash\{-1\}, \psi)\}$ is a smooth atlas on S^1, since:

$$\phi \circ \psi^{-1} : (-\pi, 0) \cup (0, \pi) \to (0, \pi) \cup (0, 2\pi), \qquad \phi \circ \psi^{-1}(\theta) = \begin{cases} \theta & \text{if } \theta \in (0, \pi), \\ \theta + 2\pi & \text{if } \theta \in (-\pi, 0), \end{cases}$$

is smooth, and similarly $\psi \circ \phi^{-1}$ is smooth. It follows that S^1 is a 1-manifold with $\{(S^1\backslash\{1\}, \phi), (S^1\backslash\{-1\}, \psi)\}$ a representative of its smooth structure.

2 New manifolds from old

Just as we described when we studied group theory, there are three basic constructions we can use to produce new mathematical objects from old ones: taking subsets, quotients and products. In the theory of manifolds, we can introduce the notion of 'submanifolds' and 'product manifolds' as you might expect, but the idea of 'quotient manifolds' is unfortunately rather complicated, and hence beyond the scope of the course.[2]

Embedded submanifolds

We would like to define an *embedded submanifold*[3] to be a subset of a manifold which is a manifold in its own right, inheriting the smooth structure of the ambient manifold in an appropriate way. The complication here is to decide what it means to 'inherit the smooth structure of the ambient manifold'.

Our definition will be based on the following idea: to define an embedded submanifold Y of dimension m inside an n-manifold X (we say that Y has *codimension* $n-m$ in this case), we ask that about each point $p \in Y$, there is a chart (U, ϕ) in some representative atlas of the smooth structure of the *ambient* manifold X (in this way, we hope to inherit the smooth structure from X) such that $Y \cap U$ is described by the restriction of ϕ to only m of its components; equivalently, we ask that $n-m$ of the components of ϕ vanish on $Y \cap U$. Let's make this precise:

Definition 2.1: Let X be an n-manifold. A subset $Y \subseteq X$ is an *embedded submanifold of codimension* k if for all $p \in Y$ there exists a chart (U, ϕ) whose domain includes p, with $\phi = (x^1, ..., x^n)$, in some representative atlas of the smooth structure of X such that:

$$Y \cap U = \{q \in U : x^1(q) = ... = x^k(q) = 0\}.$$

That is, Y is locally $(n-k)$-Euclidean, through projections of charts drawn from an atlas in the smooth structure of X.

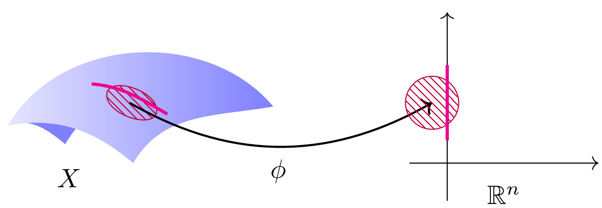

Figure 4: *About any point p in an embedded submanifold of codimension k, there is a chart drawn from a representative smooth atlas in the smooth structure of the* ambient *in which the embedded submanifold is described by the vanishing of k coordinates.*

[2] To give an indication of what can go wrong, we note that quotients of a Hausdorff space are not necessarily Hausdorff. So deciding on when a quotient of a manifold is a manifold is at least as hard as deciding when a quotient of a Hausdorff space is Hausdorff, which is a rather difficult problem!

[3] Unfortunately, the term *submanifold* on its own is already reserved in the mathematical literature for something slightly more general.

This definition does indeed result in the correct structure on Y. To prove this, it will be useful to first prove a small lemma which characterises smooth equivalence of atlases in a slightly different way:

Lemma 2.2: Let $\mathcal{A}, \mathcal{B}$ be two smooth atlases on the topological n-manifold X. Then $\mathcal{A}, \mathcal{B}$ are smoothly equivalent if and only if for any charts $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$, we have that the transition functions:

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \qquad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are smooth.

Proof: Suppose $\mathcal{A}, \mathcal{B}$ are smoothly equivalent. Let $\phi = (x^1, ..., x^n)$; then each $x^i : U \to \mathbb{R}$ is obviously a smooth function with respect to $\mathcal{A}$ (for any $(U', \phi') \in \mathcal{A}$, we have that $\phi \circ (\phi')^{-1}$ is smooth, and hence the projection of this map onto any of its components is smooth), thus each $x^i : U \to \mathbb{R}$ is a smooth function with respect to $\mathcal{B}$. In particular, $x^i \circ \psi^{-1} : \psi(U \cap V) \to \mathbb{R}$ is smooth for each i, so $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is smooth. Similarly $\psi \circ \phi^{-1}$ is smooth.

For the converse, let $f : X \to \mathbb{R}$ be smooth with respect to $\mathcal{A}$. Given any chart $(V, \psi) \in \mathcal{B}$, let $\{(U_\alpha, \phi_\alpha)\} \subseteq \mathcal{A}$ be a subset of charts drawn from $\mathcal{A}$ whose domains cover V. For any point $\psi(p) \in \psi(V)$, we have $\psi(p) \in \psi(V \cap U_\alpha)$ for some α. Thus we can write $f \circ \psi^{-1} = f \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \psi^{-1}$ on $\psi(V \cap U_\alpha)$. But $f \circ \phi_\alpha^{-1}$ is smooth, and $\phi_\alpha \circ \psi^{-1}$ is smooth by assumption, so $f \circ \psi^{-1}$ is smooth at $\psi(p)$. It follows that f is smooth with respect to $\mathcal{B}$. Similarly, if f was smooth with respect to $\mathcal{B}$, it would be smooth with respect to $\mathcal{A}$, so we're done. $\square$

We are now ready to prove the main result:

Proposition 2.3: Let X be an n-manifold, and let $Y \subseteq X$ be a submanifold of codimension k. Then Y is naturally an $(n-k)$-manifold.

Proof: Consider equipping Y with the subspace topology. Then Y inherits Hausdorfness and second-countability from X. Next, note that given $p \in Y$, by assumption there exists a chart (U, ϕ) whose domain contains p such that:

$$Y \cap U = \{q : x^1(q) = ... = x^k(q) = 0\},$$

where $\phi = (x^1, ..., x^n)$. In particular, this implies that the restriction $(Y \cap U, (x^{k+1}|_U, ..., x^n|_U))$ is a chart on $Y \cap U$. It is obviously a bijection. Furthermore, it is continuous since ϕ is continuous and the projection onto any coordinate is continuous; similarly the inverse is continuous, since the map $(x^{k+1}, ..., x^n) \to (0, ..., 0, x^{k+1}, ..., x^n)$ is continuous, and ϕ^{-1} is continuous. Therefore, Y is locally $(n-k)$-Euclidean, and hence is a topological $(n-k)$-manifold.

Let $\{(Y \cap U_\alpha, (x_\alpha^{k+1}|_{U_\alpha}, ..., x_\alpha^n|_{U_\alpha}))\}$ be the collection of all charts on Y of this form. We claim that this is a smooth atlas for Y, so can be taken as a representative of a smooth structure for Y. Certainly the domains of the charts cover Y by construction, so it remains to check that transition functions are smooth. We note that the general transition function takes the form:

$$(x_\alpha^{k+1}|_{U_\alpha}, ..., x_\alpha^n|_{U_\alpha}) \circ (x_\beta^{k+1}|_{U_\beta}, ..., x_\beta^n|_{U_\beta})^{-1}.$$

Suppose that these charts are inherited from charts $(U_\alpha, \phi_\alpha = (x_\alpha^1, ..., x_\alpha^n))$ and $(U_\beta, \phi_\beta = (x_\beta^1, ..., x_\beta^n))$ on X. These may be drawn from different representatives of the smooth structure of X, but by the lemma above, we must have $\phi_\alpha \circ \phi_\beta^{-1}$ smooth since the two representative atlases are smoothly equivalent. The above transition function is then is just pre-composed and post-composed with appropriate projections, hence it must be smooth. $\square$

A particularly useful way of producing embedded submanifolds is by identifying *open subsets* of manifolds, which are *always* embedded submanifolds:

Theorem 2.4: (**The open subset theorem**) An open subset $Y \subseteq X$ of an n-manifold X is an embedded submanifold of codimension 0. In particular, Y is an n-manifold in its own right.

Proof: Given any point $p \in Y$, let (U, ϕ) be a chart about p in some representative smooth atlas $\mathcal{A}$ in the smooth structure of X. We would like to restrict the chart to $Y \cap U$; to do so, we claim that:

$$\mathcal{A} \cup \{(Y \cap U, \phi|_{Y \cap U})\}$$

is also a smooth atlas in the smooth structure of X. To show this, suppose that f is smooth with respect to $\mathcal{A}$. Then $f \circ \phi|_{Y\cap U}^{-1} : \phi(Y \cap U) \to \mathbb{R}$ is certainly smooth, since it is simply the restriction of the smooth function $f \circ \phi^{-1}$ to $\phi(Y \cap U)$. On the other hand if f is smooth with respect to $\mathcal{A} \cup \{(Y \cap U, \phi|_{Y\cap U})\}$, then it is obviously smooth with respect to $\mathcal{A}$. Therefore, we can without loss of generality take $(Y \cap U, \phi|_{Y\cap U})$ as a chart about p. Using this chart, we have:

$$Y \cap (Y \cap U) = \{q \in Y \cap U\},$$

so by definition Y is an embedded submanifold of codimension 0. $\square$

Product manifolds

The construction of product manifolds is a little simpler, and proceeds exactly in the way you would expect:

Proposition 2.5: Let X, Y be n, m-manifolds respectively. Then $X \times Y$ is naturally an $(n + m)$-manifold.

Proof: Certainly $X \times Y$ is a Hausdorff, second-countable topological space with respect to the standard product topology. We now aim to produce a smooth atlas for $X \times Y$.

Let $\mathcal{A}, \mathcal{B}$ be representatives of the smooth structures of X, Y respectively. Define:

$$\mathcal{C} := \{(U \times V, (\phi, \psi)) : (U, \phi) \in \mathcal{A}, (V, \psi) \in \mathcal{B}\}.$$

We claim that $\mathcal{C}$ is a smooth atlas for $X \times Y$. Certainly charts of the form $(U \times V, (\phi, \psi))$ cover $X \times Y$, since charts of the form (U, ϕ), (V, ψ) cover X, Y respectively. It is also clear that $(\phi, \psi) : U \times V \to \phi(U) \times \psi(V)$ is a bijection, is continuous in the product topology (since both ϕ, ψ are individually continuous), and has continuous inverse $(\phi^{-1}, \psi^{-1}) : \phi(U) \times \psi(V) \to U \times V$ (since ϕ^{-1}, ψ^{-1} are individually continuous). Furthermore, in general transition functions take the form:

$$(\phi, \psi) \circ ((\phi')^{-1}, (\psi')^{-1}) = (\phi \circ (\phi')^{-1}, \psi \circ (\psi')^{-1}),$$

which are smooth since $\phi \circ (\phi')^{-1}, \psi \circ (\psi')^{-1}$ are individually smooth. $\square$

We will need this construction when we look at Lie groups G, because we would like to define the group operation to be a smooth map from $G \times G$ to G; in particular, we will need $G \times G$ to have some smooth structure.

3 Maps between manifolds

Now we have defined manifolds and described how to obtain new instances of manifolds from existing instances, we can begin defining natural maps between manifolds.

Smooth maps

When we initially defined manifolds, we motivated everything by the desire to define *smooth maps* $f : X \to \mathbb{R}$ on topological manifolds X. We can now reasonably upgrade our definitions to smooth maps $f : X \to Y$ between manifolds X, Y:

Definition 3.1: Let X, Y be manifolds, and let $\mathcal{A}, \mathcal{B}$ be respective representatives of their smooth structures. We say that a map $f : X \to Y$ is *smooth* if for all charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$, we have:

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)$$

is a smooth map in the sense of real multivariable calculus. One can straightforwardly prove that the smoothness of $f : X \to Y$ is independent of the representatives of the smooth structures of X, Y chosen.

The set of smooth maps from X to Y is written $C^\infty(X, Y)$. The set of smooth maps from X to $\mathbb{R}$ is written $C^\infty(X)$.

Reassuringly, this matches up completely with the old definition which only applied to smooth maps $f : X \to \mathbb{R}$, since a representative atlas for the smooth structure of $\mathbb{R}$ is simply $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$.

Smooth maps have some basic properties, which will prove useful later in the course:

Proposition 3.2: We have the following:

(i) If $f : X \to Y, g : Y \to Z$ are smooth maps between manifolds, then $g \circ f : X \to Z$ is a smooth map between manifolds.

(ii) Let (U, ϕ) be a coordinate chart in some atlas for the manifold X. Then both $\phi : U \to \phi(U)$ and $\phi^{-1} : \phi(U) \to U$ are smooth maps between manifolds (recall the open subset U is an embedded submanifold of X).

Proof: Both (i) and (ii) are straightforward and left as an exercise to the reader. □

Diffeomorphisms

As we described above, when two groups are *isomorphic*, we cannot tell them apart using only their group structure. We can define a similar notion for manifolds, called *diffeomorphism*:

Definition 3.3: Let X, Y be manifolds. A smooth map $f : X \to Y$ which possesses a smooth inverse $f^{-1} : Y \to X$ is called a *diffeomorphism* between X and Y. If there exists a diffeomorphism between X and Y, we say that X and Y are *diffeomorphic*, and we write $X \simeq Y$.

Diffeomorphism is the correct notion of 'equivalence' because there it induces a natural bijection between the smooth structures of X and Y. In particular, if $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is a representative atlas for the smooth structure of X, then $f(\mathcal{A}) := \{(f(U_\alpha), \phi_\alpha \circ f^{-1})\}$ is a representative atlas for the smooth structure of Y. Similarly, given a representative atlas for the smooth structure of Y, we can construct a natural representative atlas for the smooth structure of X.

A basic example of a diffeomorphism comes from the proposition above:

Example 3.4: Given a chart (U, ϕ) on a manifold, we have that $\phi : U \to \phi(U)$ is a diffeomorphism, since ϕ is a smooth map with a smooth inverse, by part (ii) of the proposition above.

More interestingly, but well beyond the scope of this handout, the *Whitney embedding theorem* tells us that any n-manifold X is diffeomorphic to an embedded submanifold of $\mathbb{R}^{2n}$.

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4 Connectedness and compactness

Since manifolds are topological spaces, we can check whether they have particular topological properties. In this section, we describe how *connectivity* and *compactness* properties transfer to the theory of manifolds from the theory of topological spaces.

Connectivity of manifolds

We begin by reminding the reader of some of the *connectivity* properties of a topological space:

Definition 4.1: Let X be a topological space.

- We say that X is *disconnected* if there exist open subsets $U, V \subseteq X$ such that $U \cap V = \emptyset$ (i.e. U, V are disjoint) and $U \cup V = X$ (i.e. U, V cover X). We say that X is *connected* if it is not disconnected.
- We say that X is *path-connected* if for every two points $x_1, x_2 \in X$, there exists a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.
- We say that X is *simply-connected* if it is path-connected and every continuous map $\gamma : S^1 \to X$ from the circle S^1 into the space X is *homotopic to a point*. Informally, this means that any loop in the space can be *continuously deformed to a point*. Formally, there exists a continuous map $F : S^1 \times [0, 1] \to X$ such that:

$$F(x, 0) = \gamma(x), \qquad F(x, 1) = x_1,$$

  where $x_1 \in X$ is some fixed point. We view the second parameter in F as 'time', so that at time $t = 0$, F looks like the loop γ, whilst at time $t = 1$, F looks like a constant curve.

These properties immediately transfer to manifolds; we say that a manifold X is connected if it is connected as a topological space, etc. When checking the connectivity properties of a manifold in practice, it is useful to remember the chain of implications:

simply-connected $\quad\Rightarrow\quad$ path-connected $\quad\Rightarrow\quad$ connected.

However, recall that none of these implications reverse.

For the examples of manifolds we described above, we have the following connectivity properties:

Example 4.2:

- $\mathbb{R}^n$ is simply-connected. Therefore, $\mathbb{C}^n$, $\mathrm{Mat}_n(\mathbb{R})$ and $\mathrm{Mat}_n(\mathbb{C})$ are simply-connected too.
- S^1 is path-connected, but is not simply-connected. Giving a rigorous proof that S^1 is not simply-connected requires some algebraic topology, which we consider beyond the scope of this handout. However, the reader should be fairly satisfied that, intuitively, a loop which goes around the entire circle cannot be continuously deformed to a point. On the other hand, any two points on the circle can be joined by a continuous path, so S^1 must be path-connected.

Compactness of manifolds

Another useful topological property that we can use to describe manifolds is *compactness*. This is usually defined in terms of *open covers* of topological spaces:

Definition 4.3: An *open cover* of a topological space X is a collection of open subsets of X, $\{U_\alpha\}$, whose union is X. A *subcover* $\{V_\alpha\}$ of an open cover $\{U_\alpha\}$ is an open cover such that $\{V_\alpha\} \subseteq \{U_\alpha\}$.

A topological space X is called *compact* if every open cover of X has a finite subcover.

In most cases, we deal with manifolds which are embedded in some ambient Euclidean space (this will certainly be true of the matrix Lie groups we shall describe later in the course). In this case, compactness can be characterised via two simple conditions, using the *Heine-Borel theorem*:

Theorem 4.4: (**The Heine-Borel theorem**) Let $S \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$ endowed with the subspace topology. Then S is compact if and only if:

(i) S is a closed subset of $\mathbb{R}^n$ (in the topological sense);

(ii) S is a bounded subset of $\mathbb{R}^n$; that is, there exists M such that S is fully contained in the ball of radius M, i.e. $S \subseteq \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| \leq M\}$.

Proof: The proof is technical, and is left to an undergraduate course in topology. □

Example 4.5:

- The manifold $\mathbb{R}^n$ is non-compact. Therefore, $\mathbb{C}^n$, $\mathrm{Mat}_n(\mathbb{R})$ and $\mathrm{Mat}_n(\mathbb{C})$ are also non-compact.
- S^1 is compact, since it is a closed, bounded subset of $\mathbb{R}^2$.