Lie groups

Abstract

In the previous handouts, we have prepared the way for the introduction of one of the most important objects of study of the Part III Symmetries, Fields and Particles course, namely *Lie groups*. Formally, a Lie group is an object which is both a *group* (as defined in an earlier handout) and a *manifold* (also defined in an earlier handout), where the group multiplication and group inversion are smooth maps between manifolds. This naturally codifies the idea of a 'group which can be parametrised in terms of smooth coordinates', the type of group that we have previously argued is most relevant in physical applications.

In this handout, we begin by clearly stating the definition of a Lie group and providing some basic examples. Next, we describe various Lie group constructions (namely *Lie subgroups* and the *direct product* of Lie groups), before introducing structure-preserving maps between Lie groups. In the penultimate section, we translate information about Lie groups into the language of *local coordinates* about the identities of Lie groups. Finally, we review examples of *matrix groups*, which provide a plentiful source of interesting Lie groups for the rest of the course.

1 Lie groups: definitions and examples

We begin by defining Lie groups, combining the two structures we have defined previously:

Definition 1.1: A *Lie group* is a group G which is also a manifold, such that the group multiplication $\cdot : G \times G \to G$ and group inversion $(-)^{-1} : G \to G$ are smooth maps between manifolds.

We will shortly see many interesting examples of Lie groups in the form of *matrix Lie groups*. For now, we give some very basic examples instead:

Example 1.2: $(\mathbb{R}^n, +)$ is a Lie group. Certainly $(\mathbb{R}^n, +)$ is a group, and $\mathbb{R}^n$ is an n-manifold as we saw above (with a representative atlas $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$). It remains to show that the group addition and inversion are smooth maps between Lie groups.

We note that the product manifold $\mathbb{R}^n \times \mathbb{R}^n$ has a representative atlas given by $\{(\mathbb{R}^n \times \mathbb{R}^n, (\mathrm{id}_{\mathbb{R}^n}, \mathrm{id}_{\mathbb{R}^n}))\}$, so to check that the group operation $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth, it suffices to consider:

$$(\mathrm{id}_{\mathbb{R}^n}, \mathrm{id}_{\mathbb{R}^n}) \circ + \circ (\mathrm{id}_{\mathbb{R}^n}^{-1}, \mathrm{id}_{\mathbb{R}^n}^{-1})(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1 + \mathbf{x}_2.$$

This is clearly a smooth map in the sense of multivariable calculus, so it follows that indeed the group operation is smooth. To check that group inversion $(-)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, we simply note that:

$$\mathrm{id}_{\mathbb{R}^n} \circ (-)^{-1} \circ \mathrm{id}_{\mathbb{R}^n}^{-1}(\mathbf{x}) = -\mathbf{x}$$

is a smooth map in the sense of multivariable calculus. So the group inversion is also smooth, and we're done.

From this basic example, we can produce many more related examples:

Example 1.3: $\mathbb{C}^n$, $\mathrm{Mat}_n(\mathbb{R})$ and $\mathrm{Mat}_n(\mathbb{C})$ are all Lie groups under addition.

A slightly less trivial example of a Lie group is the circle S^1, under multiplication of complex numbers:

Example 1.4: We showed in a previous handout that the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a 1-manifold. It is also a group under multiplication of complex numbers, since:

- Given $z, w \in S^1$, we have $|zw| = |z| \cdot |w| = 1 \cdot 1 = 1$, so $zw \in S^1$. It follows that S^1 is closed under multiplication.
- Multiplication of complex numbers is associative.
- $1 \in S^1$, which serves as an identity.
- The inverse of $z \in S^1$ is given by $1/z$, which is in S^1 since $|1/z| = 1/|z| = 1/1 = 1$.

To show that S^1 is a *Lie* group, it remains to show that multiplication $\cdot : S^1 \times S^1 \to S^1$ and inversion $(-)^{-1} : S^1 \to S^1$ in the group are smooth maps between manifolds. This requires a little more work since S^1 is covered by two charts rather than just one; we saw earlier that a representative of the smooth structure is given by $\{(S^1\backslash\{1\}, \phi), (S^1\backslash\{-1\}, \psi)\}$, with:

$$\phi(e^{i\theta}) = \theta, \qquad \theta \in (0, 2\pi),$$

$$\psi(e^{i\theta}) = \theta, \qquad \theta \in (-\pi, \pi).$$

To check that inversion is smooth, we should check that the four maps $\phi \circ (-)^{-1} \circ \phi^{-1}$, $\phi \circ (-)^{-1} \circ \psi^{-1}$, $\psi \circ (-)^{-1} \circ \phi^{-1}$ and $\psi \circ (-)^{-1} \circ \psi^{-1}$ are all smooth. To see this, note that:

$$(-)^{-1} \circ \phi^{-1}(\theta) = e^{-i\theta}, \qquad \theta \in (0, 2\pi),$$

$$(-)^{-1} \circ \psi^{-1}(\theta) = e^{-i\theta}, \qquad \theta \in (-\pi, \pi).$$

Then we have:

$$\phi \circ (-)^{-1} \circ \phi^{-1}(\theta) = 2\pi - \theta, \qquad \psi \circ (-)^{-1} \circ \psi^{-1}(\theta) = -\theta,$$

and:

$$\psi \circ (-)^{-1} \circ \phi^{-1}(\theta) = \begin{cases} -\theta, & \theta \in (0, \pi), \\ 2\pi - \theta, & \theta \in (\pi, 2\pi), \end{cases} \qquad \phi \circ (-)^{-1} \circ \psi^{-1}(\theta) = \begin{cases} -\theta, & \theta \in (-\pi, 0), \\ 2\pi - \theta, & \theta \in (0, \pi). \end{cases}$$

all of which are smooth maps.

The product manifold $S^1 \times S^1$ has a representative smooth atlas comprising four charts:

$$\big\{(S^1\backslash\{1\} \times S^1\backslash\{1\}, (\phi, \phi)), \quad (S^1\backslash\{1\} \times S^1\backslash\{-1\}, (\phi, \psi))$$

$$(S^1\backslash\{-1\} \times S^1\backslash\{1\}, (\psi, \phi)), \quad (S^1\backslash\{-1\} \times S^1\backslash\{-1\}, (\psi, \psi))\big\}.$$

With these charts identified, there are now eight coordinate maps we need to check for smoothness, to ensure that multiplication $\cdot : S^1 \times S^1 \to S^1$ is indeed smooth, e.g. $\phi \circ \cdot \circ (\phi, \phi)^{-1}$, $\phi \circ \cdot \circ (\phi, \psi)^{-1}$, etc. Indeed all of these are smooth; we leave this as an exercise for the (slightly masochistic) reader. It follows that indeed S^1 is a Lie group.

2 New Lie groups from old

As usual, we can construct new Lie groups from existing instances of Lie groups by taking *subsets* and *products*, then appropriately endowing them with a Lie group structure.[1]

Embedded Lie subgroups

We begin by defining *embedded Lie subgroups* in the natural way:

Definition 2.1: Let G be a Lie group. An *embedded Lie subgroup* $H \leq G$ is a subgroup which is also an embedded submanifold.

With this definition, an embedded Lie subgroup is naturally a Lie group in its own right:

Proposition 2.2: Let G be a Lie group of dimension n. An embedded Lie subgroup $H \leq G$ of codimension k is a Lie group of dimension $n-k$ in its own right.

Proof: We already have that H is a group, since it is a subgroup, and it is a manifold of dimension $n-k$, since it is an embedded submanifold of codimension k. Therefore, we need only show that the group operation and group inversion on H are smooth functions between manifolds.

Recall that we can form a representative atlas on H of the form $\{(H \cap U_\alpha, (x_\alpha^{k+1}|_{U_\alpha}, ..., x_\alpha^n|_{U_\alpha}))\}$, where $(U_\alpha, (x_\alpha^1, ..., x_\alpha^n))$ are charts on G satisfying:

$$H \cap U_\alpha = \{g \in U_\alpha : x_\alpha^1(g) = ... = x_\alpha^k(g) = 0\}.$$

In particular, the general coordinate expression for group inversion takes the form:

$$(x_\alpha^{k+1}|_{U_\alpha}, ..., x_\alpha^n|_{U_\alpha}) \circ (-)^{-1} \circ (x_\beta^{k+1}|_{U_\beta}, ..., x_\beta^n|_{U_\beta})^{-1}.$$

Since G is a Lie group, we have that $(x_\alpha^1, ..., x_\alpha^n) \circ (-)^{-1} \circ (x_\beta^1, ..., x_\beta^n)^{-1}$ is a smooth map; the above map is just this map pre-composed and post-composed with appropriate projections, so must itself be smooth. Similarly multiplication in H is smooth. $\square$

In practice, a very useful tool in discovering embedded Lie subgroups is Cartan's *closed subgroup theorem*. We will use this theorem frequently when we discuss matrix Lie groups in the next section, but the proof is rather involved and is considered beyond the scope of the course.

Theorem 2.3: (***Cartan's closed subgroup theorem***) Let G be a Lie group. A subgroup $H \leq G$ is an embedded Lie subgroup if and only if H is closed (in the topological sense, rather than the group-theoretic sense).

Proof: Beyond the scope of the course. $\square$

[1] Quotients are beyond the scope of the course; see the discussion of quotient manifolds in a previous handout.

Example 2.4: We can check that $\mathbb{R}^{n-1} = \{(x_1, .., x_{n-1}, 0) : x_i \in \mathbb{R}, i = 1, .., n-1\} \leq \mathbb{R}^n$ is an embedded Lie subgroup in two ways:

- Certainly $\mathbb{R}^{n-1}$ is a subgroup. It is also closed, since sequences with elements of the form $(x_1, ..., x_{n-1}, 0)$ necessarily tend to limits of the same form. Thus $\mathbb{R}^{n-1}$ is an embedded Lie subgroup by the closed subgroup theorem.
- On the other hand, instead of checking the closure of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$, we could check it is an embedded submanifold. Given any point $(p_1, ..., p_{n-1}, 0) \in \mathbb{R}^{n-1}$, we have that $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is a chart about $(p_1, ..., p_{n-1}, 0)$ such that:

$$\mathbb{R}^n \cap \mathbb{R}^{n-1} = \{\mathbf{x} : x_n = 0\}.$$

  Therefore, $\mathbb{R}^{n-1}$ is indeed an embedded submanifold of $\mathbb{R}^n$, of codimension 1.

Direct products of Lie groups

The direct product of Lie groups also forms a Lie group in the natural way.

Proposition 2.5: Let G, H be Lie groups. The Cartesian product $G \times H$ is a Lie group under the standard direct product group structure, and the standard product manifold structure.

Proof: We just need to check that the group multiplication and group inversion are smooth maps between manifolds. If $\mathcal{A}, \mathcal{B}$ are representative smooth atlases on G, H respectively, recall that:

$$\mathcal{C} := \{(U \times V, (\phi, \psi)) : (U, \phi) \in \mathcal{A}, (V, \psi) \in \mathcal{B}\}$$

is a representative smooth atlas on the product manifold $G \times H$. To check that inversion is smooth, note that a general coordinate representation of $(-)^{-1} : G \times H \to G \times H$ with respect to this atlas is given by:

$$(\phi_1, \psi_1) \circ (-)^{-1} \circ (\phi_2, \psi_2)^{-1} = (\phi_1 \circ (-)^{-1} \circ \phi_2^{-1}, \psi_1 \circ (-)^{-1} \circ \psi_2^{-1}).$$

Both components on the right are smooth functions, so the left is a smooth function. Similarly for multiplication. □

3 Maps between Lie groups

In this brief section, we quickly describe the natural maps between Lie groups. The basic maps are defined in precisely the way you would expect:

Definition 3.1: Let G, H be Lie groups. A *Lie group homomorphism* from G to H is a map $\theta : G \to H$ which is both *smooth* (when viewed as a map between manifolds) and a *homomorphism* (when viewed as a map between groups). In the special case where $G = H$, we call θ a *Lie group endomorphism*.

Definition 3.2: Let G, H be Lie groups. A *Lie group isomorphism* from G to H is a bijective Lie group homomorphism $\theta : G \to H$. In the special case where $G = H$, we call θ a *Lie group automorphism*.

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4 Local coordinates on Lie groups

Whilst the above abstract formalism is very nice, it is sometimes useful to have a coordinate description of the Lie group in a neighbourhood of the identity. We can achieve this as follows.

Let G be a Lie group, and let (U, ϕ) be a chart whose domain contains the identity e. Without loss of generality, we may assume the chart is *centred* at the identity e, so that $\phi(e) = \mathbf{0}$; this is an immediate consequence of the following proposition.

Proposition 4.1: Let X be a manifold, and let $\mathcal{A}$ be a representative of its smooth structure. If (U, ϕ) is a chart whose domain contains the point $p \in X$, then $(U, \phi - \phi(p))$ is a chart centred at p, and:

$$\mathcal{A}' = \mathcal{A} \cup \{(U, \phi - \phi(p))\}$$

is also a representative of the smooth structure on X.

Proof: First, note that if $\phi : U \to \phi(U)$ is a homeomorphism, then $\phi - \phi(p) : U \to (\phi(U) - \phi(p))$ is also a homeomorphism, since translations are homeomorphisms and the composition of homeomorphisms is a homeomorphism. Thus $(U, \phi - \phi(p))$ is indeed a chart. Furthermore, $(\phi - \phi(p))(p) = \phi(p) - \phi(p) = \mathbf{0}$, so this chart is indeed centred at p.

It remains to show that $\mathcal{A}'$ is a representative of the smooth structure on X. Clearly if $f : X \to \mathbb{R}$ is smooth with respect to $\mathcal{A}'$, then f is smooth with respect to $\mathcal{A}$. On the other hand, if $f : X \to \mathbb{R}$ is smooth with respect to $\mathcal{A}$, then:

$$f \circ (\phi - \phi(p))^{-1} : (\phi(U) - \phi(p)) \to \mathbb{R}$$

is the composition $f \circ (\phi - \phi(p))^{-1} = f \circ \phi^{-1} \circ ((-) + \phi(p))$, where $(-) + \phi(p) : \phi(U) - \phi(p) \to \phi(U)$ is a translation so is smooth, and $f \circ \phi^{-1}$ is smooth by assumption. In particular, it is smooth; thus $\mathcal{A}, \mathcal{A}'$ are smoothly equivalent, and the result follows. $\square$

Thus, we have some local coordinates ϕ on a neighbourhood U of the identity of our Lie group, centred at the identity. To more clearly display this, let's write $g(\mathbf{x})$ for the unique element of U whose local coordinates are $\mathbf{x} \in \phi(U)$, i.e. $\phi(g(\mathbf{x})) = \mathbf{x}$. Since we assume $\phi(e) = \mathbf{0}$, we have $g(\mathbf{0}) = e$.

We have now parametrised the group in a neighbourhood of the identity, U, in terms of some coordinates, $\mathbf{x} \in \phi(U)$. We might naturally ask: can we describe the group multiplication law in terms of these coordinates? Or: can we write down the coordinates of the inverse of the group element $g(\mathbf{x}) \in U$? The answer to both these questions is yes, *provided* that we are sufficiently close to the identity.

To see why there is a problem, suppose that $g(\mathbf{x}), g(\mathbf{y}) \in U$. In order to obtain a 'coordinate description' of the group multiplication, we would like to write $g(\mathbf{x})g(\mathbf{y}) = g(\mu(\mathbf{x}, \mathbf{y}))$ for some coordinates $\mu(\mathbf{x}, \mathbf{y}) \in \phi(U)$. But this is *only* a well-defined procedure if the product $g(\mathbf{x})g(\mathbf{y})$ remains in U, so that it can be labelled by some coordinates $\mu(\mathbf{x}, \mathbf{y}) \in \phi(U)$. Similar considerations apply when taking the inverse.

As alluded to above though, we can avoid all problems by working very close to the identity:

Proposition 4.2: Let G be a Lie group, and let (U, ϕ) be a chart centred on the identity $e \in G$. Write $g(\mathbf{x})$ for the element of U with coordinates $\mathbf{x}$, i.e. the unique element satisfying $\phi(g(\mathbf{x})) = \mathbf{x}$.

For any integer $n \geq 2$, there exists a sequence of open neighbourhoods of the identity, $V_1 \subseteq V_2 \subseteq ... \subseteq V_{n-1} \subseteq V_n = U$, such that:

(i) For all $g(\mathbf{x}) \in V_i$, we have $g(\mathbf{x})^{-1} \in V_{i+1}$, for $i = 1, ..., n-1$.

(ii) For all $g(\mathbf{x}), g(\mathbf{y}) \in V_i$, we have $g(\mathbf{x})g(\mathbf{y}) \in V_{i+1}$, for $i = 1, ..., n-1$.

Proof: The construction is inductive. Suppose we have already constructed $V_{i+1}, V_{i+2}, ..., V_n = U$ with the desired properties. We need to construct V_i such that whenever $g(\mathbf{x}), g(\mathbf{y}) \in V_i$, we have $g(\mathbf{x})^{-1} \in V_{i+1}$ and $g(\mathbf{x})g(\mathbf{y}) \in V_{i+1}$.

Recall that the group multiplication, say $m : G \times G \to G$, is a smooth map between manifolds, so is in particular a continuous map between topological spaces. Since V_{i+1} is an open neighbourhood of the identity, it follows that $m^{-1}(V_{i+1})$ is an open neighbourhood of (e, e) in $G \times G$, since $m(e, e) = e$. By definition of the product topology, it follows that we can write:

$$m^{-1}(V_{i+1}) = \bigcup_{\alpha} U_\alpha^1 \times U_\alpha^2$$

for some open subsets $U_\alpha^1, U_\alpha^2 \subseteq G$ with at least one α^* such that $(e, e) \in U_{\alpha^*}^1 \times U_{\alpha^*}^2$. Fix such an α^*, and define $V_i' = U_{\alpha^*}^1 \cap U_{\alpha^*}^2$. Then V_i' is open since it is the finite intersection of open sets, and $e \in V_i'$ since $e \in U_{\alpha^*}^1, e \in U_{\alpha^*}^2$. Note also that $U_{\alpha^*}^1 \subseteq V_{i+1}$, since $U_{\alpha^*}^1 = m(U_{\alpha^*}^1 \times \{e\}) \subseteq V_{i+1}$; hence $V_i' \subseteq V_{i+1}$. Finally, $V_i' \times V_i' \subseteq m^{-1}(V_{i+1})$.

Similarly, recall that the group inversion, say $i : G \to G$, is a smooth map between manifolds, so is in particular a continuous map between topological spaces. Since V_{i+1} is an open neighbourhood of the identity, it follows that $i^{-1}(V_{i+1})$ is an open neighbourhood of e, since $i(e) = e$. Let us define:

$$V_i = i^{-1}(V_{i+1}) \cap V_i'.$$

Then V_i is open since it is the finite intersection of open sets, contains the identity since $e \in V_i'$ and $e \in i^{-1}(V_{i+1})$, and is such that $i(V_i) \subseteq V_{i+1}$ and $m(V_i \times V_i) \subseteq V_{i+1}$. Finally, $V_i \subseteq V_{i+1}$ since $V_i' \subseteq V_{i+1}$. The result follows. □

In particular, it follows that we can write:

$$g(\mathbf{x})g(\mathbf{y}) = g(\mu_i(\mathbf{x}, \mathbf{y})), \qquad g(\mathbf{x})^{-1} = g(\iota_i(\mathbf{x})),$$

for some collection of maps $\mu_i : \phi(V_i) \times \phi(V_i) \to \phi(V_{i+1})$, $\iota_i : \phi(V_i) \to \phi(V_{i+1})$. This describes the group multiplication law in local coordinates about the identity, as desired.

From now on, we will drop most of the notation, and simply say 'for sufficiently small $\mathbf{x}$' whenever we need $\mathbf{x}$ to lie in V_i for some i (e.g. if we need the product of three elements, we need to work with three nested open neighbourhoods $V_1 \subseteq V_2 \subseteq V_3 = U$, with the coordinates of all three elements lying in V_1). For notational convenience, we drop the index on μ_i, ι_i and leave it implied.

Each of the group properties (identity, inverses, and associativity), and the fact that we are working with a *Lie group*, contributes some structure to the maps μ, ι:

Proposition 4.3: We have the following properties of the maps μ, ι:

(i) μ, ι are smooth maps. More strongly, μ is an *analytic* function at the point $(\mathbf{0}, \mathbf{0})$, and ι is an analytic function at the point $\mathbf{0}$; that is, about these points, μ, ι are *equal* to their Taylor series.

(ii) For sufficiently small $\mathbf{x}, \mathbf{y}$, we have $\mu(\mathbf{x}, \mathbf{0}) = \mathbf{x}$ and $\mu(\mathbf{0}, \mathbf{y}) = \mathbf{y}$.

(iii) For sufficiently small $\mathbf{x}$, we have $\mu(\mathbf{x}, \iota(\mathbf{x})) = \mu(\iota(\mathbf{x}), \mathbf{x}) = \mathbf{0}$.

(iv) For sufficiently small $\mathbf{x}, \mathbf{y}, \mathbf{z}$, we have $\mu(\mathbf{x}, \mu(\mathbf{y}, \mathbf{z})) = \mu(\mu(\mathbf{x}, \mathbf{y}), \mathbf{z})$.

Proof: (i) Smoothness follows immediately from smoothness of group multiplication and inversion as maps between Lie groups. Explicitly, we recall that by definition:

$$\mu = (\phi, \phi) \circ \cdot \circ (\phi, \phi)^{-1}, \qquad \iota = \phi \circ (-)^{-1} \circ \phi^{-1},$$

so these are smooth by definition of a Lie group. Analyticity is much harder, and considered beyond the scope of the course.

You will prove (ii), (iii) and (iv) as part of the exercises for this handout. They follow from the axioms of identity, inverses and associativity respectively. $\square$

5 Matrix Lie groups

A plentiful sources of Lie groups comes in the form of *matrix Lie groups*.[2] The basic example is $GL(n, \mathbb{C})$.

Definition 5.1: The *general linear group* $GL(n, \mathbb{F})$ over the field $\mathbb{F}$ is the set of $n \times n$ invertible matrices with entries in $\mathbb{F}$. In this course, we always take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Proposition 5.2: $GL(n, \mathbb{C})$ is a Lie group of dimension $2n^2$.

Proof $GL(n, \mathbb{C})$ is a group: Given $A, B \in GL(n, \mathbb{C})$, we have $\det(AB) = \det(A)\det(B) \neq 0$, since A, B are invertible. Thus AB is invertible, and it follows that $GL(n, \mathbb{C})$ is closed. Next, note that matrix multiplication is associative, so the axiom of associativity is satisfied. Note that the identity matrix is in $GL(n, \mathbb{C})$ since $\det(I) = 1$, and hence the axiom of identity is satisfied. Finally, if $A \in GL(n, \mathbb{C})$, we have that A^{-1} is invertible with inverse A, thus $A^{-1} \in GL(n, \mathbb{C})$. It follows that indeed $GL(n, \mathbb{C})$ is a group.

Proof $GL(n, \mathbb{C})$ is a Lie group: Note that the determinant $\det : \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{C}$ is a continuous map. Therefore, $\det^{-1}(\mathbb{C}\backslash\{0\}) = GL(n, \mathbb{C})$ is an open subset of $\mathrm{Mat}_n(\mathbb{C})$. It follows by the open subset theorem that $GL(n, \mathbb{C})$ is a $2n^2$-manifold. Furthermore, we know from the proof of the open subset theorem that a representative atlas on $GL(n, \mathbb{C})$ is inherited from $\mathrm{Mat}_n(\mathbb{C})$ simply by restricting the domains of charts on $\mathrm{Mat}_n(\mathbb{C})$. In particular, a *global* chart on $GL(n, \mathbb{C})$ is given by $\phi : GL(n, \mathbb{C}) \to \phi(GL(n, \mathbb{C}))$ with:

$$\phi \begin{pmatrix} x_{11} + iy_{11} & x_{12} + iy_{12} & \cdots & x_{1n} + iy_{1n} \\ x_{21} + iy_{21} & x_{22} + iy_{22} & \cdots & x_{2n} + iy_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} + iy_{n1} & x_{n2} + iy_{n2} & \cdots & x_{nn} + iy_{nn} \end{pmatrix} = \begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{nn} \\ x_{11} \\ x_{12} \\ \vdots \\ x_{nn} \end{pmatrix}, \tag{$*$}$$

with $x_{ij}, y_{ij} \in \mathbb{R}$. It follows immediately that the group inversion and group multiplication are smooth maps between manifolds in this case, since both inversion and multiplication simply involve taking smooth functions of the real and imaginary parts of the matrix entries. So we're done. $\square$

The closed subgroup theorem now allows us to produce many Lie groups from $GL(n, \mathbb{C})$:

Definition 5.3: A *matrix Lie group* is a (topologically) closed subgroup of $GL(n, \mathbb{C})$. Equivalently, by the closed subgroup theorem, any matrix Lie group is an embedded Lie subgroup of $GL(n, \mathbb{C})$.

Most straightforwardly, $GL(n, \mathbb{R})$ is a matrix Lie group, since it is an closed subgroup of $GL(n, \mathbb{C})$:

Proposition 5.4: The general linear group $GL(n, \mathbb{R})$ is an embedded Lie subgroup of $GL(n, \mathbb{C})$; in particular, it is a Lie group in its own right. It has dimension n^2.

[2] Though note that not all Lie groups are matrix Lie groups.

Proof: Certainly $GL(n, \mathbb{R})$ is a group, by the same argument we used for $GL(n, \mathbb{C})$ above. It is also topologically closed, since if $A_i \in GL(n, \mathbb{R})$ is a sequence in $GL(n, \mathbb{R})$ whose limit is $A \in GL(n, \mathbb{C})$, then all the entries of A must be real (since taking real parts is a continuous function). Hence the limit satisfies $A \in GL(n, \mathbb{R})$. It follows by the closed subgroup theorem that $GL(n, \mathbb{C})$ is an embedded Lie subgroup of $GL(n, \mathbb{C})$.

Unfortunately, the dimension of $GL(n, \mathbb{R})$ is not provided by the closed subgroup theorem. We can instead compute the dimension directly by constructing a chart on an open subset of $GL(n, \mathbb{R})$. The global chart $(GL(n, \mathbb{C}), \phi)$ described in $(*)$ above immediately restricts to an appropriate chart, since:

$$GL(n, \mathbb{R}) \cap GL(n, \mathbb{C}) = \{q \in GL(n, \mathbb{C}) : \phi_1(q) = ... = \phi_{n^2}(q) = 0\},$$

where $\phi = (\phi_1, ..., \phi_{2n^2})$. Thus $GL(n, \mathbb{R})$ has dimension $2n^2 - n^2 = n^2$. $\square$

With the general linear groups $GL(n, \mathbb{C}), GL(n, \mathbb{R})$ defined, and proved to be Lie groups of dimension $2n^2$, n^2 respectively, we can introduce some more interesting matrix groups. This will be the focus of the rest of this section.

Special linear groups

Restricting to matrices which have unit determinant, i.e. those which preserve volumes, we obtain the *special linear group*:

Definition 5.5: The *special linear group* $SL(n, \mathbb{F})$ over the field $\mathbb{F}$ is the subset of matrices in $GL(n, \mathbb{F})$ which have unit determinant.

Proposition 5.6: $SL(n, \mathbb{C})$ is a Lie group of dimension $2n^2 - 2$, and $SL(n, \mathbb{R})$ is a Lie group of dimension $n^2 - 1$.

Proof: First, we show that $SL(n, \mathbb{F})$ is a subgroup of $GL(n, \mathbb{F})$. Certainly $I \in SL(n, \mathbb{F})$, since I is invertible and $\det(I) = 1$; therefore, $SL(n, \mathbb{F})$ is non-empty. Now let $A, B \in SL(n, \mathbb{F})$. Then $\det(AB^{-1}) = \det(A)/\det(B) = 1$, since $\det(A) = \det(B) = 1$ for $A, B \in SL(n, \mathbb{F})$. It follows that $AB^{-1} \in SL(n, \mathbb{F})$; thus by the subgroup test, we have $SL(n, \mathbb{F}) \leq GL(n, \mathbb{F})$.

To show that $SL(n, \mathbb{F})$ is topologically closed in $GL(n, \mathbb{F})$, let $A_i \in SL(n, \mathbb{F})$ be a sequence in $SL(n, \mathbb{F})$, with limit $A \in GL(n, \mathbb{F})$. Since $\det$ is a continuous function, we have $1 = \det(A_i) \to \det(A)$, thus $\det(A) = 1$, and it follows that the limit is also in $SL(n, \mathbb{F})$. Thus $SL(n, \mathbb{F})$ is topologically closed in $GL(n, \mathbb{F})$; therefore, by the closed subgroup theorem, we have that $SL(n, \mathbb{F})$ is an embedded Lie subgroup of $GL(n, \mathbb{F})$.

Again, the closed subgroup theorem does not immediately give us the dimension of $SL(n, \mathbb{F})$. To find it, we should exhibit some chart on $SL(n, \mathbb{F})$. Recall that we already have a global chart on $GL(n, \mathbb{C})$, given by $(*)$ above. We claim that a compatible chart is given by:

$$\psi \left(Z = \begin{pmatrix} x_{11} + iy_{11} & x_{12} + iy_{12} & \cdots & x_{1n} + iy_{1n} \\ x_{21} + iy_{21} & x_{22} + iy_{22} & \cdots & x_{2n} + iy_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} + iy_{n1} & x_{n2} + iy_{n2} & \cdots & x_{nn} + iy_{nn} \end{pmatrix} \right) = \begin{pmatrix} \mathrm{Im}(\det(Z) - 1) \\ \mathrm{Re}(\det(Z) - 1) \\ y_{12} \\ \vdots \\ y_{nn} \\ x_{12} \\ \vdots \\ x_{nn} \end{pmatrix}.$$

This is a smooth map in the sense of multivariable calculus, since the real and imaginary parts of $\det(Z) - 1$ are smooth functions of the real and imaginary parts of the entries of Z; therefore, this map must be invertible, at least in some small neighbourhood of the identity. Furthermore, $\psi \circ \phi^{-1}$ is obviously smooth in the sense of multivariable calculus, where ϕ is the global chart on $GL(n, \mathbb{C})$ given by $(*)$ above. It follows that ψ can be considered a chart in a compatible smooth atlas on $GL(n, \mathbb{C})$. Further, we note:

$$SL(n, \mathbb{C}) \cap GL(n, \mathbb{C}) = \{q \in GL(n, \mathbb{C}) : \psi_1(q) = \psi_2(q) = 0\}$$

and

$$SL(n, \mathbb{R}) \cap GL(n, \mathbb{C}) = \{q \in GL(n, \mathbb{C}) : \psi_1(q) = \psi_2(q) = \ldots = \psi_{n^2+1}(q) = 0\}.$$

The dimensions of $SL(n, \mathbb{C})$ and $SL(n, \mathbb{R})$ as Lie groups follow. $\square$

Orthogonal groups

The orthogonal groups provide a further interesting example of real matrix Lie groups.

Definition 5.7: The *orthogonal group* $O(n)$ is the set of all $n \times n$ real matrices A satisfying $A^T A = I$. The *special orthogonal group* $SO(n)$ is the set of all $n \times n$ real matrices A satisfying *both* $A^T A = I$ and $\det(A) = 1$.

Recall that orthogonal matrices are interesting because they are the unique matrices which preserve the Euclidean norm. In particular, they effect *rigid-body transformations*, namely rotations and reflections, in Euclidean space. For matrices in $SO(n)$, the condition that the determinant be positive implies that the transformations they effect are additionally *orientation preserving*; in particular, $SO(n)$ consists of all the rotations, and $O(n) \backslash SO(n)$ consists of all the reflections.

Naturally, both the orthogonal and special orthogonal groups are Lie groups:

Proposition 5.8: The orthogonal group $O(n)$ and special orthogonal group $SO(n)$ are embedded Lie subgroups of $GL(n, \mathbb{R})$. In particular, they are Lie groups in their own right. We have $\dim(O(n)) = \frac{1}{2}n(n-1)$ and $\dim(SO(n)) = \frac{1}{2}n(n-1)$.

Proof: The proofs that $O(n), SO(n)$ are groups is left as an exercise to the reader. To see that $O(n)$ is topologically closed in $GL(n, \mathbb{R})$, note that given a sequence $A_n \in O(n)$ with limit $A \in GL(n, \mathbb{R})$ we have that $I = A_n^T A_n \to A^T A$ since $A_n \mapsto A_n^T A_n$ is a continuous function; therefore $A^T A = I$, and it follows that $O(n)$ is topologically closed. Similarly $SO(n)$ is topologically closed in $GL(n, \mathbb{R})$. Thus by the closed subgroup theorem, we have that $O(n), SO(n)$ are Lie groups.

Computing the dimensions of $O(n), SO(n)$ is left as an exercise to the reader at the end of the handout; again, it can be done by exhibiting charts (this time it is useful to consider a chart on $GL(n, \mathbb{R})$ with entries given by all possible inner products of the columns of the preimage matrix). However, it is worth pondering why these groups have the *same* dimension, even though $SO(n)$ ostensibly has an additional 'condition' imposed upon its elements of having unit determinant. □

Unitary groups

Unitary groups are the complex analogues of orthogonal groups.

Definition 5.9: The *unitary group* $U(n)$ is the set of all $n \times n$ real matrices A satisfying $A^\dagger A = I$. The *special unitary group* $SU(n)$ is the set of all $n \times n$ real matrices A satisfying *both* $A^\dagger A = I$ and $\det(A) = 1$.

Similarly, $U(n)$ and $SU(n)$ are Lie groups:

Proposition 5.10: The unitary group $U(n)$ and special unitary group $SU(n)$ are embedded Lie subgroups of $GL(n, \mathbb{C})$. In particular, they are Lie groups in their own right. We have $\dim(U(n)) = n^2$ and $\dim(SU(n)) = n^2 - 1$.

Proof: The proof is similar to $O(n), SO(n)$ and is left as an exercise to the reader. In particular, the proof of the dimensionality of the groups is left as an exercise at the end of the handout. □

✎ Exercises

A (∗) denotes a more difficult exercise.

Local coordinates on Lie groups

1. Let (U, ϕ) be a chart on a Lie group whose domain is centred on the identity e. Write the element of U with coordinates $\mathbf{x} \in \phi(U)$. In a sufficiently small neighbourhood of the identity, write:

$$g(\mathbf{x})g(\mathbf{y}) = g(\mu(\mathbf{x}, \mathbf{y})), \qquad g(\mathbf{x})^{-1} = g(\iota(\mathbf{x})),$$

where μ is an analytic function at $(\mathbf{0}, \mathbf{0})$, and ι is an analytic function at $\mathbf{0}$.

   (a) Why must $\mu(\mathbf{x}, \mathbf{0}) = \mathbf{x}$ and $\mu(\mathbf{0}, \mathbf{y}) = \mathbf{y}$? Why must $\mu(\mathbf{x}, \iota(\mathbf{x})) = \mu(\iota(\mathbf{x}), \mathbf{x}) = \mathbf{0}$?

   (b) Explain why the Taylor series of $\mu(\mathbf{x}, \mathbf{y})$ about $(\mathbf{0}, \mathbf{0})$ takes the form:

$$\mu^a(\mathbf{x}, \mathbf{y}) = x^a + y^a + c^a{}_{bc} x^b y^c + O(x^2 y, xy^2) \qquad (\dagger)$$

   for some coefficients $c^a{}_{bc}$. Determine the Taylor series of $\iota(\mathbf{x})$ up to order $O(x^3)$ in terms of $c^a{}_{bc}$.

   (c) Explain why $\mu(\mathbf{x}, \mu(\mathbf{y}, \mathbf{z})) = \mu(\mu(\mathbf{x}, \mathbf{y}), \mathbf{z})$. (∗) Using an expansion to **one higher order** than (†), prove the *Jacobi identity*:

$$f^r{}_{st} f^s{}_{uv} + f^r{}_{su} f^s{}_{vt} + f^r{}_{sv} f^s{}_{tu} = 0,$$

   where $f^a{}_{bc} := c^a{}_{bc} - c^a{}_{cb}$.

Matrix Lie groups

2. (a) Prove that $SO(2)$ is an Abelian group.

   (b) Is $O(2)$ Abelian or non-Abelian? If the latter, exhibit two $O(2)$ matrices that do not commute.

3. Check that the orthogonal group $O(n)$ and the unitary group $U(n)$ are groups. Verify that $O(n)$ and $SO(n)$ are the subgroups of real matrices in $U(n)$ and $SU(n)$ respectively. By considering the action of $U(n)$ on $\mathbb{C}^n$, and identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, show that $U(n)$ is a subgroup of $O(2n)$. (∗) Show further that $U(n)$ is a subgroup of $SO(2n)$.

4. (a) Show that $SO(n)$ is a normal subgroup of $O(n)$.

   (b) If n is odd, show that $\mathbb{Z}_2 \cong \{I_n, -I_n\}$ is a normal subgroup of $O(n)$, where I_n is the $n \times n$ identity matrix. Prove that $O(n)$ can be written as a direct product, $O(n) \cong SO(n) \times \mathbb{Z}_2$.

   (c) Explain why the results of (b) do not hold for n even. (∗) Prove that when n is even, we can write $O(n)$ as a semi-direct product, $O(n) \cong SO(n) \rtimes_\phi \mathbb{Z}_2$, for some homomorphism $\phi : \mathbb{Z}_2 \to \mathrm{Aut}(SO(n))$ to be determined.

5. Suppose that we construct a matrix $M \in O(n)$ column by column, from left to right. Explain why the first column is an arbitrary unit vector, the second is a unit vector orthogonal to the first column, ..., the kth column is a unit vector orthogonal to the span of the first $k-1$ columns, etc. Deduce the dimension of $O(n)$ as a Lie group. By similar reasoning, determine the dimension of $U(n)$ as a Lie group.

6. (a) Show that a general element of $SU(2)$ can be written as:

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

   where α, β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

   (b) Deduce that an alternative form for an $SU(2)$ matrix is

$$U = a_0 I + i\mathbf{a} \cdot \boldsymbol{\sigma},$$

   with $(a_0, \mathbf{a})$ real, $\boldsymbol{\sigma}$ the vector of Pauli matrices, and $a_0^2 + |\mathbf{a}|^2 = 1$.

   (c) Using the second form, calculate the product of two $SU(2)$ matrices.