Symmetries, Fields and Particles
Examples Class 4

Supervisor: James Moore
Email: jmm232@cam.ac.uk.

Part 4 of the course: Semisimple, complex Lie algebras $\mathfrak{g}$

(1) Understand **structure** of $\mathfrak{g}$. This involves constructing all the Lie brackets of $\mathfrak{g}$ in some privileged bases (called *Cartan-Weyl* bases and *Chevalley* bases).

(2) **Classify** all possible structures. This is achieved by examining the structure of the *roots* of the Lie algebra by studying *Cartan matrices* and *Dynkin diagrams*.

(3) Understand **representations** of $\mathfrak{g}$. The representation theory is an application of the structure theory in (1) and (2). *(We won't have time to discuss this, but I encourage you to read the notes of Dexter Chua, Josh Kirklin, and Jan Gutowski for further information.)*

1: Lie bracket structure of $\mathfrak{g}$

Basic definitions

A *Lie algebra* over the field $\mathbb{F}$ is a vector space $\mathfrak{g}$ equipped with a *Lie bracket* $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

$$[x,x] = 0, \qquad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

for all $x, y, z \in \mathfrak{g}$. A Lie algebra is *Abelian* if $[x,y] = 0$ for all $x, y \in \mathfrak{g}$.

An *isomorphism* between Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ is a bijective linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ preserving the Lie bracket: $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}_1$. An isomorphism is an *automorphism* if $\mathfrak{g}_1 = \mathfrak{g}_2$.

Basic definitions

A *subalgebra* is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ which is closed under the Lie bracket. We write $\mathfrak{h} \leq \mathfrak{g}$. An *ideal* is a subalgebra $\mathfrak{h} \leq \mathfrak{g}$ which is *strongly closed* under the Lie bracket, i.e. for all $x \in \mathfrak{g}$, $h \in \mathfrak{h}$, we have $[x, h] \in \mathfrak{h}$. We write $\mathfrak{h} \trianglelefteq \mathfrak{g}$.

A Lie algebra is *simple* if $\mathfrak{g}$ has no non-trivial proper ideals. A Lie algebra is *semisimple* if $\mathfrak{g}$ has no non-trivial Abelian ideals.

From now on: Everything we say will apply in general only to *finite-dimensional, complex, semisimple Lie algebras* $\mathfrak{g}$.

Cartan subalgebras

A *Cartan subalgebra* is a subalgebra $\mathfrak{h} \leq \mathfrak{g}$ satisfying:

(i) $\mathfrak{h}$ is Abelian;

(ii) $\mathfrak{h}$ is *ad-diagonalisable*: for all $h \in \mathfrak{h}$, we have that the map $\mathrm{ad}(h) : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}(h)x = [h, x]$ is diagonalisable;

(iii) $\mathfrak{h}$ is *maximal*: if $x \in \mathfrak{g}$ is such that $[x, h] = 0$ for all $h \in \mathfrak{h}$, we have $x \in \mathfrak{h}$.

Key idea: maximum amount of simultaneous diagonalisation! Since all $h, h' \in \mathfrak{h}$ commute, we have $[\mathrm{ad}(h), \mathrm{ad}(h')] = \mathrm{ad}([h, h']) = 0$, so can simultaneously diagonalise all $\mathrm{ad}(h)$.

Facts of life about Cartan subalgebras

For finite-dimensional, complex, semisimple Lie algebras $\mathfrak{g}$, we have that:

(1) Cartan subalgebras always exist.

(2) Two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$ are always equivalent, in the sense that there always exists an automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ satisfying $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$.

In particular, all Cartan subalgebras have the same dimension. We call the dimension of any Cartan subalgebra the *rank* of the Lie algebra $\mathfrak{g}$.

Roots and root spaces

Given a choice of Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$, since the adjoint maps $\mathrm{ad}(h)$ are all diagonalisable and are all commuting, their are all simultaneously diagonalisable. Let x be a simultaneous eigenvector of $\mathrm{ad}(h)$. Then for all $h \in \mathfrak{h}$:

$$\mathrm{ad}(h)x = [h, x] = \alpha(h)x$$

for some complex h-dependent eigenvalues $\alpha(h)$.

Note that α depends linearly on h, so α is a map from $\mathfrak{h}$ to $\mathbb{C}$. Therefore the eigenvalue is a *functional* $\alpha : \mathfrak{h} \to \mathbb{C}$, so lives in the *dual space* to $\mathfrak{h}$.

Roots and root spaces

There are two possibilities for the eigenvalue $\alpha \in \mathfrak{h}^*$:

- $\alpha \equiv 0$. Then $\mathrm{ad}(h)x = [h, x] = 0$ for all $h \in \mathfrak{h}$, and it follows that $x \in \mathfrak{h}\backslash\{0\}$ by maximality of the Cartan subalgebra. Conversely, if $x \in \mathfrak{h}\backslash\{0\}$ we have $\mathrm{ad}(h)x = [h, x] = 0$ since the Cartan subalgebra is Abelian, and hence x is a simultaneous eigenvector of all $\mathrm{ad}(h)$ with eigenvalue 0.
- $\alpha \not\equiv 0$. In this case we say that $\alpha \in \mathfrak{h}^*\backslash\{0\}$ is a *root* of the Lie algebra with respect to the Cartan subalgebra $\mathfrak{h}$. The set of all roots is finite (there can only be finitely many eigenvalues!) and is called the *root system* with respect to the Cartan subalgebra $\mathfrak{h}$, written $R(\mathfrak{h})$.

Roots and root spaces

The eigenspaces associated with the roots are called *root spaces*. Explicitly, the eigenspace associated with the root α is:

$$\mathfrak{g}_\alpha(\mathfrak{h}) = \{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Sometimes it is useful to talk about *generalised root spaces*, where $\mathfrak{g}_\alpha(\mathfrak{h})$ is defined as above, but α is allowed to be any functional in $\mathfrak{h}^*$. Then $\mathfrak{g}_0(\mathfrak{h}) = \mathfrak{h}$, and $\mathfrak{g}_\alpha(\mathfrak{h}) = 0$ when α is not a root.

The Cartan decomposition

The above work allows us to decompose $\mathfrak{g}$ into simultaneous eigenspaces of $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{h})} \mathfrak{g}_\alpha(\mathfrak{h}).$$

This is called the *Cartan decomposition*. The Lie brackets are given by:

- $[h, h'] = 0$ for all $h, h' \in \mathfrak{h}$;
- $[h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha(\mathfrak{h})$;
- $[x, y] = ...?$ for all $x \in \mathfrak{g}_\alpha(\mathfrak{h}), y \in \mathfrak{g}_\beta(\mathfrak{h})$.

The Cartan decomposition

For the final bracket $[x, y]$ with $x \in \mathfrak{g}_\alpha(\mathfrak{h})$ and $y \in \mathfrak{g}_\beta(\mathfrak{h})$, we can at least work out where the result lands. Using the Jacobi identity, we have for all $h \in \mathfrak{h}$:

$$[h, [x, y]] = [x, [h, y]] - [y, [h, x]] = \beta(h)[x, y] - \alpha(h)[y, x] = (\alpha+\beta)(h)[x, y].$$

Hence $[x, y] \in \mathfrak{g}_{\alpha+\beta}(\mathfrak{h})$, where $\mathfrak{g}_{\alpha+\beta}(\mathfrak{h})$ is a *generalised* root space. In particular:

- $[x, y] \in \mathfrak{h}$ if $\alpha + \beta = 0$.
- $[x, y]$ is in the root space $\mathfrak{g}_{\alpha+\beta}(\mathfrak{h})$ if $\alpha + \beta$ is a root.
- $[x, y] = 0$ otherwise.

The Killing form and the case $\alpha + \beta = 0$

Fact: The Killing form restricts to a symmetric, non-degenerate bilinear form on the Cartan subalgebra, $\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$. (Proof: Easy using Cartan's theorem.)

This induces a canonical isomorphism $\theta : \mathfrak{h} \to \mathfrak{h}^*$ given by $\theta(h) = \kappa(h, -)$. In particular, this implies that to every root α we can associate an element of the Cartan subalgebra via $H_\alpha = \theta^{-1}(\alpha)$.

Using non-degeneracy of the Killing form, it can then be shown that for all $x \in \mathfrak{g}_\alpha(\mathfrak{h}), y \in \mathfrak{g}_{-\alpha}(\mathfrak{h})$:

$$[x, y] = \kappa(x, y) H_\alpha.$$

Another fact of life about root spaces

We now have all the brackets:

$$[h, h'] = 0, \qquad [h, x] = \alpha(h)x, \qquad [x, y] = \begin{cases} \kappa(x, y)H_\alpha & \text{if } \alpha + \beta = 0 \\ z \in \mathfrak{g}_{\alpha+\beta}(\mathfrak{h}) & \text{if } \alpha + \beta \text{ is a root} \\ 0 & \text{otherwise.} \end{cases}$$

To finish off, we need another fact about root spaces.

Fact: All roots are non-degenerate, i.e. $\dim(\mathfrak{g}_\alpha(\mathfrak{h})) = 1$ for all root spaces. (Proof: Hard! Use representations of $\mathfrak{su}_{\mathbb{C}}(2)$ on the Lie algebra.)

Cartan-Weyl bases

Overall, this implies we can define a basis for the Lie algebra as follows:

- Choose a Cartan subalgebra $\mathfrak{h}$ and choose a basis $\{H_1, ..., H_r\}$.
- For each root $\alpha \in R(\mathfrak{h})$, choose a generator E_α for the corresponding one-dimensional space $\mathfrak{g}_\alpha(\mathfrak{h})$.
- For each root, define its *components* wrt $\{H_1, ..., H_r\}$ as $\alpha_i = \alpha(H_i)$.

Together, $\{H_1, ..., H_r\} \cup \bigcup \{E_\alpha\}$ form a basis for the Lie algebra called a *Cartan-Weyl basis*. Note in particular,

$$\dim(\mathfrak{g}) = \operatorname{rank}(\mathfrak{g}) + |R(\mathfrak{h})|.$$

Cartan-Weyl bases

The Lie brackets in this privileged basis are given by:

$$[H_i, H_j] = 0, \qquad [H_i, E_\alpha] = \alpha_i E_\alpha,$$

$$[E_\alpha, E_\beta] = \begin{cases} \kappa(E_\alpha, E_{-\alpha}) H_\alpha & \text{if } \alpha + \beta = 0, \\ N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root,} \\ 0 & \text{otherwise.} \end{cases}$$

This leaves only some proportionality constants $N_{\alpha\beta}$ to determine.

Brief detour: $\mathfrak{su}_{\mathbb{C}}(2)$ subalgebras

The above Lie bracket structure is actually hiding some subalgebras isomorphic to $\mathfrak{su}_{\mathbb{C}}(2)$. We have the following fact:

Fact: If α is a root, then $-\alpha$ is a root. Furthermore, $k\alpha$ is a root for some $k \in \mathbb{C}$ if and only if $k = \pm 1$. (Proof: First part easy using non-degeneracy of Killing form. Second part hard; requires considering representations of $\mathfrak{su}_{\mathbb{C}}(2)$ on $\mathfrak{g}$.)

This fact allows us to identify $\{H_\alpha, E_{\pm\alpha}\}$ as generating an $\mathfrak{su}_{\mathbb{C}}(2)$ subalgebra.

Brief detour: $\mathfrak{su}_{\mathbb{C}}(2)$ subalgebras

We can see how $\{H_\alpha, E_{\pm\alpha}\}$ generate an $\mathfrak{su}_{\mathbb{C}}(2)$ subalgebra by examining the brackets:

$$[H_\alpha, E_{\pm\alpha}] = \pm\alpha(H_\alpha)E_{\pm\alpha} = \pm\theta(\theta^{-1}(\alpha))(H_\alpha)E_{\pm\alpha} = \pm\kappa(H_\alpha, H_\alpha)E_{\pm\alpha},$$

and

$$[E_\alpha, E_{-\alpha}] = \kappa(E_\alpha, E_{-\alpha})H_\alpha.$$

We see that scaling $H_\alpha, E_{\pm\alpha}$, we can create an $\mathfrak{su}_{\mathbb{C}}(2)$ subalgebra. We have:

$$h_\alpha = \frac{2H_\alpha}{\kappa(H_\alpha, H_\alpha)}, \qquad e_{\pm\alpha} = \frac{\sqrt{2}E_{\pm\alpha}}{\sqrt{\kappa(H_\alpha, H_\alpha)\kappa(E_\alpha, E_{-\alpha})}}$$

Brief detour: $\mathfrak{su}_{\mathbb{C}}(2)$ subalgebras

It can be shown that $\kappa(H_\alpha, H_\alpha) \neq 0$ and $\kappa(E_\alpha, E_{-\alpha}) \neq 0$ allowing this rescaling to take place. Under this rescaling, the relations become:

$$[h_\alpha, e_{\pm\alpha}] = \pm 2 e_{\pm\alpha}, \qquad [e_\alpha, e_{-\alpha}] = h_\alpha,$$

which are exactly the brackets of an $\mathfrak{su}_{\mathbb{C}}(2)$ algebra.

Note: h_α is completely fixed by this construction, but $e_{\pm\alpha}$ can still be rescaled again, as long as we keep $\kappa(e_\alpha, e_{-\alpha}) = 1$. This means we are free to scale $e_\alpha \mapsto \eta_\alpha e_\alpha, e_{-\alpha} \mapsto e_{-\alpha}/\eta_\alpha$ and keep this subalgebra the same.

Another brief detour: a bilinear form on the roots

As we mentioned earlier, the Killing form κ is a non-degenerate symmetric bilinear form when restricted to $\mathfrak{h} \times \mathfrak{h}$. We can use this to define a non-degenerate symmetric bilinear form on the roots by setting:

$$(\alpha, \beta) = \kappa(\theta^{-1}(\alpha), \theta^{-1}(\beta)) = \kappa(H_\alpha, H_\beta),$$

where $\theta : \mathfrak{h} \to \mathfrak{h}^*$ is the canonical isomorphism $\theta(h) = \kappa(h, -)$. In particular, for the basis defined above, we have:

$$[h_\alpha, e_\beta] = \beta(h_\alpha)e_\beta = \theta(\theta^{-1}(\beta))(h_\alpha)e_\beta = \frac{2\kappa(H_\beta, H_\alpha)}{\kappa(H_\alpha, H_\alpha)}e_\beta = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}e_\beta.$$

Determination of proportionality constants $N_{\alpha\beta}$

In general, determination of the constants $N_{\alpha\beta}$ is very difficult, as they satisfy some non-trivial relations. For example, using the Jacobi identity, you saw in the lectures that we can deduce relations of the form:

$$[e_\alpha,[e_{-\alpha},e_{\beta-n\alpha}]]+[e_{-\alpha},[e_{\beta-n\alpha},e_\alpha]]=-[e_{\beta-n\alpha},[e_\alpha,e_{-\alpha}]]$$

$$\Rightarrow \qquad N_{\beta-n\alpha,-\alpha}N_{\beta-(n-1)\alpha,\alpha}-N_{\beta-(n+1)\alpha,-\alpha}N_{\beta-n\alpha,\alpha}=\frac{2(\alpha,\beta-n\alpha)}{(\alpha,\alpha)}.$$

Summing, we can infer information about the products of the constants $N_{\alpha\beta}$ (in particular, if we sum up from $n=1$ to $n=p$ where p is the largest integer such that $\beta-p\alpha$ is a root, the sum telescopes, and $N_{\beta-(p+1)\alpha,-\alpha}$ drops out).

Determination of proportionality constants $N_{\alpha\beta}$

This information is coupled with a particular *normalisation* called *Chevalley normalisation*: we demand that the constants obey $N_{\alpha,\beta} = N_{-\alpha,-\beta}$. It turns out that this can always be guaranteed, by choosing the generators e_α of the root spaces such that the linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ extended from:

$$\sigma(h) = -h, \qquad \sigma(e_\alpha) = -e_{-\alpha}, \qquad \text{where } h \in \mathfrak{h},$$

is an automorphism of the Lie algebra. This corresponds to taking the Hermitian conjugate in a matrix Lie algebra (which are the only things we care about in physics), with generators replaced by $h \mapsto ih$, $e_\alpha \mapsto ie_\alpha$.

Determination of proportionality constants $N_{\alpha\beta}$

It turns out that Chevalley normalisation $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ is enough to determine the constants (up to signs) together with the relations from the Jacobi identity (though the proof is very technical). The result is:

$$[e_\alpha, e_\beta] = \begin{cases} h_\alpha & \text{if } \alpha+\beta = 0 \\ \pm(p+1)e_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ 0 & \text{otherwise,} \end{cases}$$

where p is the largest integer such that $\beta - p\alpha$ is a root.

Determination of proportionality constants $N_{\alpha\beta}$

The signs in $N_{\alpha\beta} = \pm(p+1)$ are quite difficult to pin down too. It turns out that certain signs are free and can be chosen by changing generators; these signs correspond to pairs of roots (α, β) called *extraspecial pairs*. The other signs are fixed once we have chosen the signs for the extraspecial root pairs.

Chevalley bases

To construct a final basis with all brackets determined then, we do the following:

- Let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra, and let the subset of the roots $\{\alpha_{(1)}, ..., \alpha_{(r)}\}$ form a basis for $\mathfrak{h}^*$ (note since the roots span, a subset are linearly independent). By the canonical isomorphism $\theta : \mathfrak{h} \to \mathfrak{h}^*$ induced by the Killing form, we have that the normalised elements

$$h_i := h_{\alpha_{(i)}} = \frac{2\theta^{-1}(\alpha_{(i)})}{\kappa(\theta^{-1}(\alpha_{(i)}), \theta^{-1}(\alpha_{(i)}))} = \frac{2H_{\alpha_{(i)}}}{\kappa(H_{\alpha_{(i)}}, H_{\alpha_{(i)}})}$$

  form a basis for $\mathfrak{h}$.

Chevalley bases

- Let $\{h_1, ..., h_r\}$ be the basis for $\mathfrak{h}$ defined as above. Let e_α be normalised generators of the root spaces $\mathfrak{g}_\alpha(\mathfrak{h})$, satisfying $\kappa(e_\alpha, e_{-\alpha}) = 1$, and such that the linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ defined on a basis by:

  $$\sigma(h_i) = -h_i, \qquad \sigma(e_{\pm\alpha}) = -e_{\mp\alpha},$$

  is an automorphism.
- Choose signs for all extraspecial root pairs (α, β).

We have that $\{h_1, ..., h_r\} \cup \bigcup\{e_\alpha\}$ forms a basis called a *Chevalley basis* for the Lie algebra.

Chevalley bases

In a Chevalley basis, all brackets of the Lie algebra are given by:

$$[h_i, h_j] = 0, \qquad [h_i, e_\alpha] = \frac{2(\alpha_{(i)}, \alpha)}{(\alpha_{(i)}, \alpha_{(i)})} e_\alpha,$$

$$[e_\alpha, e_\beta] = \begin{cases} h_\alpha & \text{if } \alpha + \beta = 0 \\ \pm(p+1) e_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ 0 & \text{otherwise,} \end{cases}$$

where p is the greatest integer such that $\beta - p\alpha$ is a root.

What have we achieved?

To summarise:

- Given a choice of Cartan subalgebra $\mathfrak{h}$, we have completely determined all Lie brackets in the Lie algebra (up to some sign convention).
- All of these Lie brackets are determined in terms of properties of the roots. In particular, if we know all the roots, we would like to understand the ratio:

  $$\frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

  for each pair of roots α, β; these are the only non-trivial structure constants of the Lie algebra. By linearity it's sufficient to know these ratios on a basis.

Question 1

A Lie algebra has Cartan subalgebra $\underline{H} = (H_1, ..., H_r)$ and the remaining generators are $E_{\underline{\alpha}}$, corresponding to roots $\underline{\alpha}$, where $[\underline{H}, E_{\underline{\alpha}}] = \underline{\alpha} E_{\underline{\alpha}}$. Assume $[E_{\underline{\alpha}}, E_{-\underline{\alpha}}] = H_{\underline{\alpha}} = 2\underline{\alpha} \cdot \underline{H}/\underline{\alpha}^2$. For a root $\underline{\beta}$, $E_{\underline{\beta}}$ satisfies:

$$[E_{\underline{\alpha}}, E_{\underline{\beta}}] = 0, \qquad [H_{\underline{\alpha}}, E_{\underline{\beta}}] = nE_{\underline{\beta}}, \qquad \underbrace{[E_{-\underline{\alpha}}, [\ldots, [E_{-\underline{\alpha}}}_{r \text{ times}}, E_{\underline{\beta}}], \ldots] = E_{\underline{\beta}-r\underline{\alpha}}.$$

(a) Show that

$$[E_{\underline{\alpha}}, E_{\underline{\beta}-r\underline{\alpha}}] = r(n-r+1)E_{\underline{\beta}-(r-1)\underline{\alpha}}.$$

(b) For n an integer, show that we may assume $E_{\underline{\beta}-(n+1)\underline{\alpha}} = 0$.

Notation is different! Unfortunately, this is just the way the theory is - everyone has their own notation.

What is everything? We note:

- $\{H_1, ..., H_r\}$ is a basis of the Cartan subalgebra.
- $E_{\underline{\alpha}}$ is a generator for the $\underline{\alpha}$-root space, since $[\underline{H}, E_{\underline{\alpha}}] = \underline{\alpha} E_{\underline{\alpha}}$.
- The generators are normalised such that $[E_{\underline{\alpha}}, E_{-\underline{\alpha}}] = H_{\underline{\alpha}}$, which we see corresponds to h_α in our earlier notation.
- The relation $[E_{\underline{\alpha}}, E_{\underline{\beta}}] = 0$ tells us that $\underline{\alpha} + \underline{\beta}$ is not a root.

- The relation $[H_{\underline{\alpha}}, E_{\underline{\beta}}] = nE_{\underline{\beta}}$ tells us about the product on roots:

$$[H_{\underline{\alpha}}, E_{\underline{\beta}}] = \underline{\beta}(H_{\underline{\alpha}})E_{\underline{\beta}} = \frac{2\underline{\alpha}\cdot\underline{\beta}}{\underline{\alpha}^2}E_{\underline{\beta}}.$$

  So we see that $n = 2\underline{\alpha}\cdot\underline{\beta}/\underline{\alpha}^2$.

- We are *defining* generators $E_{\underline{\beta}-r\underline{\alpha}}$ by the final relation:

$$E_{\underline{\beta}-r\underline{\alpha}} := \underbrace{[E_{-\underline{\alpha}}, [\dots, [E_{-\underline{\alpha}}}_{r\text{ times}}, E_{\underline{\beta}}], \dots].$$

  These obviously generator the $\underline{\beta} - r\underline{\alpha}$-root spaces by the theory of where the brackets land (provided $\underline{\beta} - r\underline{\alpha}$ is still a root!).

(a) Show that

$$[E_{\underline{\alpha}}, E_{\underline{\beta}-r\underline{\alpha}}] = r(n-r+1)E_{\underline{\beta}-(r-1)\underline{\alpha}}.$$

We use induction. The base case $r = 0$ is trivial, since $[E_{\underline{\alpha}}, E_{\underline{\beta}}] = 0$. Now assume the result holds for $r = k$, and consider the case $r = k+1$. We have:

$$\begin{aligned}
[E_{\underline{\alpha}}, E_{\underline{\beta}-(k+1)\underline{\alpha}}] &= [E_{\underline{\alpha}}, [E_{-\underline{\alpha}}, E_{\underline{\beta}-k\underline{\alpha}}]] \\
&= -[E_{-\underline{\alpha}}, [E_{\underline{\beta}-k\underline{\alpha}}, E_{\underline{\alpha}}]] - [E_{\underline{\beta}-k\underline{\alpha}}, [E_{\underline{\alpha}}, E_{-\underline{\alpha}}]]
\end{aligned}$$

$$
\begin{aligned}
&= k(n-k+1)[E_{-\underline{\alpha}}, E_{\underline{\beta}-(k-1)\underline{\alpha}}] - \frac{2\underline{\alpha}}{\underline{\alpha}^2} \cdot [E_{\underline{\beta}-k\underline{\alpha}}, \underline{H}] \\
&= k(n-k+1)E_{\underline{\beta}-k\underline{\alpha}} + \frac{2\underline{\alpha}}{\underline{\alpha}^2} \cdot (\underline{\beta} - k\underline{\alpha})E_{\underline{\beta}-k\underline{\alpha}} \\
&= k(n-k+1)E_{\underline{\beta}-k\underline{\alpha}} + (n-2k)E_{\underline{\beta}-k\underline{\alpha}} \\
&= (k+1)(n-k)E_{\underline{\beta}-k\underline{\alpha}}. \qquad \square
\end{aligned}
$$

(b) For n an integer, show that we may assume $E_{\underline{\beta}-(n+1)\underline{\alpha}} = 0$.

The Lie algebra is finite-dimensional, so there are only finitely many generators! Hence we must stop producing them at some point via the definition in the question.

If you're not convinced, there's another argument in the solutions which is based on $\mathfrak{su}_{\mathbb{C}}(2)$ representations. The algebra we construct in this question is an irreducible representation of $\mathfrak{su}_{\mathbb{C}}(2)$ on a finite space, which is necessary finite-dimensional (though again the reason is finite-dimensions!).

2: Classification of complex, semisimple Lie algebras

Properties of the roots

Everything we have done so far means that we can always reduce everything there is to know about a finite-dimensional, semisimple, complex Lie algebra into the questions:

- What are the roots of the Lie algebra (once we have picked a Cartan subalgebra $\mathfrak{h}$)?
- What are the ratios $\dfrac{2(\alpha, \beta)}{(\alpha, \alpha)}$ for any two roots α, β (in a basis)?

We now develop some machinery to answer these questions.

Properties of the roots

Most importantly, we have the result:

The quantisation condition: Let α, β be distinct roots. There exist integers $n_- \leq 0 \leq n_+$ such that $\beta + n\alpha$ is a root for all $n_- \leq n \leq n_+$, and where n_+ and n_- satisfy:

$$n_+ + n_- = -\frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$

We call the collection of roots $\beta + n_-\alpha$, ..., $\beta + n_+\alpha$ the *α-root string passing through β*.

Real geometry of the roots

The proof of the quantisation condition isn't trivial; to show it we consider an appropriate representation of $\mathfrak{su}_{\mathbb{C}}(2)$ on the Lie algebra. It is very important though, because it can be used to prove that the roots can be viewed as having a *real*, Euclidean geometry.

Pick a basis $\{\alpha_{(1)}, ..., \alpha_{(r)}\}$ of $\mathfrak{h}^*$ from the roots. We define:

$$\mathfrak{h}^*_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{\alpha_{(1)}, ..., \alpha_{(r)}\}.$$

It is possible to use the quantisation condition to show that $\mathfrak{h}^*_{\mathbb{R}}$ is independent of the basis we chose; call it the *real subalgebra* of the dual of the Cartan subalgebra.

Real geometry of the roots

Again, the quantisation is important in showing the following amazing result:

The symmetric, non-degenerate, bilinear form $(-,-) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ restricts to a positive definite inner product on $\mathfrak{h}^*_{\mathbb{R}} \times \mathfrak{h}^*_{\mathbb{R}}$.

This means that we can embed the roots in a real space, $\mathbb{R}^r$, and consider their geometry! In particular, we can define *lengths* and *angles* between roots:

$$|\alpha| = \sqrt{(\alpha,\alpha)}, \qquad \cos(\theta_{\alpha\beta}) = \frac{(\alpha,\beta)}{|\alpha||\beta|}.$$

Real geometry of the roots

Finally, quantisation delivers us strong constraints on the angles between the roots. Since we have:

$$\frac{2|\beta|}{|\alpha|}\cos(\theta_{\alpha\beta}) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}, \qquad \frac{2|\alpha|}{|\beta|}\cos(\theta_{\alpha\beta}) = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z},$$

it follows that $4\cos^2(\theta_{\alpha\beta}) \in \mathbb{Z}$. Remarkably, the angles between roots are constrained to take only a small set of values!

Simple roots

So far, when we've wanted to pick a basis, we've simply said 'choose a basis of roots'. However, there is a particularly convenient choice of basis which we can use the real geometry of the roots to construct.

Since $-\alpha$ is a root whenever α is a root, we can use a hyperplane through the origin to split the roots into two groups. Choose such a hyperplane, and label one group of roots as 'positive' and the other group as 'negative'.

We define a *simple root* as a positive root which cannot be written as the sum of two positive roots.

Simple roots

Simple roots indeed provide a basis, together with lots of useful properties:

- Any set of simple roots forms a basis for $\mathfrak{h}_{\mathbb{R}}^*$ (and also for $\mathfrak{h}^*$ when we allow complex linear combinations).
- Any positive root can be written as the *positive integer* linear combination of simple roots; this implies that *any* root can be written as the *integer* linear combination of simple roots.
- If α, β are simple roots, then $\alpha - \beta$ is *not* a root.

Simple roots

Further, we can generate *any* root of the Lie algebra from the simple roots via the following simple algorithm. Let Δ be a set of simple roots. Then:

Generating all roots from simple roots:

(1) Let $S = \Delta$.

(2) For each pair of *distinct* roots $\alpha \in \Delta, \beta \in S$, determine the maximum and minimum integers $n_- \leq 0 \leq n_+$ such that $\beta + n_-\alpha$ and $\beta + n_+\alpha$ are roots.

For all integers n such that $n_- \leq n \leq n_+$, we have that $\beta + n\alpha$ is a root; add all of these roots to the set S.

A useful result we can apply here are the fact that the integers n_-, n_+ obey the quantisation condition:

$$n_- + n_+ = -\frac{2(\alpha, \beta)}{(\beta, \beta)}.$$

It is also useful to remember that: (i) twice a root of the Lie algebra is never a root; (ii) if α and β are simple roots, then $\beta - \alpha$ is not a root.

(3) Repeat (2) until we generate no new elements. At this point S contains all positive roots; the full root set is given by $S \cup (-S)$.

Hence we can write all roots as integer linear combinations of the simple roots, and there's a simple algorithm telling us how to do so!

The Cartan matrix

Recall that the point of studying the properties of the roots was to better understand the ratio (on a basis):

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)},$$

Now we have a useful basis of simple roots $\{\alpha_{(1)},\alpha_{(2)},...,\alpha_{(r)}\}$, we define:

$$A_{ij} := \frac{2(\alpha_{(i)},\alpha_{(j)})}{(\alpha_{(j)},\alpha_{(j)})}.$$

This is called the *Cartan matrix*.

The Cartan matrix

Defining the Cartan matrix only on bases of simple roots gives it nice enough properties that we can *classify* all possible Cartan matrices. The way to do this is to use *Dynkin diagrams*. We define a Dynkin diagram as follows:

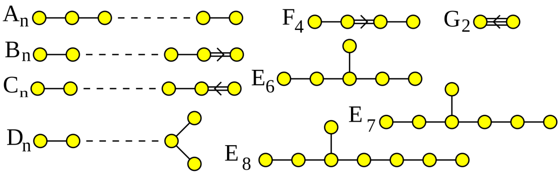
- Draw blobs for each simple root $\alpha_{(i)}$.
- Connect the blobs $\alpha_{(i)}$ and $\alpha_{(j)}$ by $A_{ij}A_{ji}$ lines.
- Draw an arrow on the lines from $\alpha_{(i)}$ to $\alpha_{(j)}$ if the root $\alpha_{(i)}$ is longer (in the sense of its length $|\alpha_{(i)}|$) than the root $\alpha_{(j)}$.

The Cartan classification

We now have the following major theorems:

- **Theorem:** The Cartan matrix of a finite-dimensional, semisimple, complex Lie algebra is independent of the choice of Cartan subalgebra, and the choice of positive roots (up to reordering of columns and rows).
- **Theorem:** The Cartan matrix uniquely determines all brackets of a finite-dimensional, semisimple, complex Lie algebra.
- **Theorem:** There is a bijection between the set of Cartan matrices of finite-dimensional, semisimple, complex Lie algebras and the set of Dynkin diagrams.

- **Theorem:** The Dynkin diagram of a finite-dimensional, *simple*, complex Lie algebra must be of one of the following types:



These theorems classify all finite-dimensional semisimple complex Lie algebras. For semisimple Lie algebras, the Dynkin diagrams are disconnected with all disconnected pieces of the above simple forms.

Question 2

A Lie algebra has simple roots $\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_r$.

(b) A rank two Lie algebra has simple roots $\boldsymbol{\alpha}_1 = (1,0)$ and $\boldsymbol{\alpha}_2 = (-1,1)$. What is the Cartan matrix?

(c) Assuming any other positive roots are equal in length to either one of the simple roots, show that $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_4 = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ are the other positive roots.

(d) Draw the root diagram, and show that the dimension of the Lie algebra is ten.

(b) We have to be careful, because the Cartan matrix is defined by:

$$A_{ij} = \frac{2(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j)}.$$

In particular, the inner products $(-, -)$ are *not* just dot products - they have something to do with the Killing form as we described above.

Throughout these questions, we assume that the components of the simple roots have been written with respect to some basis of the Cartan subalgebra $\mathfrak{h}$ such that the Killing form is diagonal with all diagonal entries 1, i.e. the Killing form becomes a *Euclidean* inner product. It is possible to show that such a basis always exists.

Making this implicit assumption, we can evaluate the Cartan matrix easily:

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

using $\boldsymbol{\alpha}_1^2 = 1, \boldsymbol{\alpha}_2^2 = 2$ and $\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 = -1$.

(c) Next, we are asked to generate all roots from the simple roots. We use the algorithm described above. Start with the set of simple roots $\Delta = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$. Setting $S = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$, we have:

- Examine roots of the form $\boldsymbol{\alpha}_2 + n\boldsymbol{\alpha}_1$. We have that $\boldsymbol{\alpha}_2$ is a root, and $\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1$ is not a root, since $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ are simple roots. Hence $n_- = 0$. In particular, it follows that:

  $$n_+ = -\frac{2\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2}{\boldsymbol{\alpha}_1^2} = 2.$$

  Hence we have that $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ are also roots which must be added to the set S.

- The set S is now given by $S = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\}$. Considering roots of the form $s + n\boldsymbol{\alpha}_1$ for $s \in S$, i.e. roots of the form $\boldsymbol{\alpha}_2 + n\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + n\boldsymbol{\alpha}_1$ and $2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + n\boldsymbol{\alpha}_1$, will not give us anything new, so we should instead start considering roots of $s + n\boldsymbol{\alpha}_2$.

  First, consider roots of type $\boldsymbol{\alpha}_1 + n\boldsymbol{\alpha}_2$. Here, $\boldsymbol{\alpha}_1$ is a root, but $\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2$ is not a root, hence $n_- = 0$. It follows that:

$$n_+ = -\frac{2\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2}{\boldsymbol{\alpha}_2^2} = 1.$$

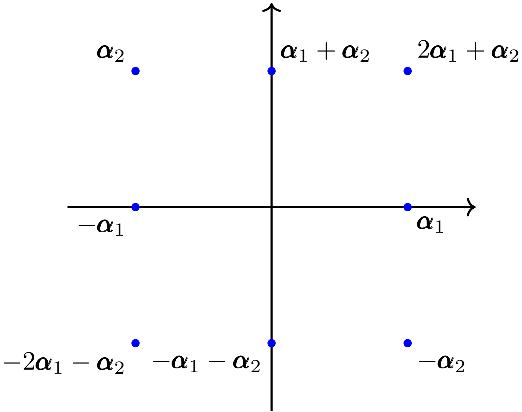
  Hence we generate the roots $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ which are already in S.

Considering roots of type $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + n\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_1 + (n+1)\boldsymbol{\alpha}_2$ will not give us anything new, and hence we are left considering roots of type $2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + n\boldsymbol{\alpha}_2 = 2\boldsymbol{\alpha}_1 + (n+1)\boldsymbol{\alpha}_2$. When $n = 0$ we have a root, but when $n = -1$ we have that $2\boldsymbol{\alpha}_1$ is not a root because it is twice a root of a Lie algebra. Hence we have $n_- = 0$. Furthermore, when $n = 1$ we have that $2\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 = 2(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)$ is not a root because it is twice a root of a Lie algebra. Hence we have $n_+ = 0$, and it follows that the only root generated this way is $2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$.

At this point, we have generated no new roots, and so the algorithm terminates. It follows that the complete root set is:

$$S = \{\pm\boldsymbol{\alpha}_1, \pm\boldsymbol{\alpha}_2, \pm(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2), \pm(2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)\}.$$

(d)


$\boldsymbol{\alpha}_2$
$\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$
$2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$
$-\boldsymbol{\alpha}_1$
$\boldsymbol{\alpha}_1$
$-2\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2$
$-\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2$
$-\boldsymbol{\alpha}_2$

If we had used a basis for $\mathfrak{h}$ where $(-,-)$ didn't act as a Euclidean inner product on the components of the simple roots, then we would have found a 'stretched' or 'distorted' square instead.

The dimension of the Lie algebra is given by the size of the root system, 8, plus the dimension of the Cartan subalgebra, 2 (which is equal to the number of simple roots, which form a basis for the dual of the Cartan subalgebra), hence is 10.

Question 3

The Lie algebra of $U(n)$ may be represented by a basis consisting of the $n^2 - n$ off diagonal matrices $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for $i \neq j$ and also the n diagonal matrices $(h_i)_{kl} = \delta_{ik}\delta_{kl}$, no sum on k, where $i, j, k, l = 1, \cdots, n$. For $SU(n)$ it is necessary to restrict to traceless matrices given by $h_i - h_j$ for some i, j. The $n - 1$ independent $h_i - h_j$ correspond to the Cartan subalgebra.

(a) Show that

$$[h_i, E_{jk}] = (\delta_{ij} - \delta_{ik})E_{jk}, \quad [E_{ij}, E_{ji}] = h_i - h_j.$$

Solution to (a): The proof of these Lie brackets is via a standard argument with index notation (see solutions).

(b) Let $\mathbf{e}_i$ be orthogonal n-dimensional unit vectors, $(\mathbf{e}_i)_j = \delta_{ij}$. Show that E_{ij} is associated with the root vector $\mathbf{e}_i - \mathbf{e}_j$ while E_{ji} corresponds to the root vector $\mathbf{e}_j - \mathbf{e}_i$.

Solution to (b): From part (a), we have:

$$[h_k, E_{ij}] = (\delta_{ki} - \delta_{kj})E_{ij} = (\mathbf{e}_i - \mathbf{e}_j)_k E_{ij}.$$

This implies that the action of the root $\mathbf{e}_i - \mathbf{e}_j \in \mathfrak{h}^*$ on a basis is $(\mathbf{e}_i - \mathbf{e}_j)(h_k) = (\mathbf{e}_i - \mathbf{e}_j)_k$. This implies that the roots in $SU(n)$ are also of this form, since $h_k - h_l$ is just a linear combination of the Cartan generators for $U(n)$; we've 'embedded' the Cartan subalgebra for $SU(n)$ into the Cartan subalgebra for $U(n)$ to make our lives easier. (**Careful:** $U(n)$ isn't semisimple, so results *only* apply to $SU(n)$.)

(c) Hence show that there are $n(n-1)$ root vectors belonging to the $n-1$ dimensional hyperplane orthogonal to $\sum_i \mathbf{e}_i$.

Solution to part (c): There's a root vector associated to every E_{ij} for i, j distinct. There are n options for i, and that leaves $n-1$ options for j, which is $n(n-1)$ in total.

Note we have:

$$(\mathbf{e}_i - \mathbf{e}_j) \cdot \left(\sum_i \mathbf{e}_i \right) = 1 - 1 = 0,$$

so the root vectors are indeed in the $(n-1)$-dimensional hyperplane normal to $\sum_i \mathbf{e}_i$.

(d) Verify that we may take as simple roots

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2,\ \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \cdots, \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \cdots, \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n,$$

by showing that all roots may be expressed in terms of α_i with either positive or negative integer coefficients.

Solution to part (d): Divide the roots into two sets by declaring $\mathbf{e}_i - \mathbf{e}_j$ to be a positive root if and only if $i < j$, and a negative root otherwise. Then α_1, ..., α_n are all positive roots. Furthermore, suppose we can write α_i as the sum of two positive roots:

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_j = \mathbf{e}_k - \mathbf{e}_l + \mathbf{e}_m - \mathbf{e}_n.$$

The conditions $i < j, k < l$ and $m < n$ are enough to force $\mathbf{e}_k = \mathbf{e}_l$ or $\mathbf{e}_m = \mathbf{e}_n$. Thus no α_i can be written as the sum of two positive roots, and it follows that α_i is a simple root for all i (with this choice of positive and negative roots).

Furthermore, for any positive root $\mathbf{e}_i - \mathbf{e}_j$, we have:

$$\mathbf{e}_i - \mathbf{e}_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j,$$

since the sum on the right hand side telescopes. Thus we can write any positive root as the positive integer linear combination of simple roots (verifying some of the general theory from earlier). Reversing the signs, we can write any negative root as the negative integer linear combination of simple roots.

(e) Determine the Cartan matrix and write down the corresponding Dynkin diagram.

Solution to (e): Again, assume the Killing form is Euclidean. Then:

$$(\alpha_i, \alpha_j) = (\mathbf{e}_i - \mathbf{e}_{i+1}, \mathbf{e}_j - \mathbf{e}_{j+1}) = \delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j} + \delta_{i+1,j+1}.$$

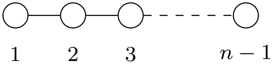
This implies that the only non-zero brackets are:

$$(\alpha_i, \alpha_i) = 2, \qquad (\alpha_i, \alpha_{i+1}) = -1, \qquad (\alpha_{i+1}, \alpha_i) = -1.$$

Inserting this information into the formula for the Cartan matrix, we have:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

To finish, we note that the only non-zero product between (distinct) simple roots is $(\alpha_i, \alpha_{i+1}) = -1$, so we should insert one line between each node in our Dynkin diagram. Thus the Dynkin diagram is of the form:



This verifies the simplest Dynkin diagram we saw in the classification theorem earlier!

Question 4

The Lie algebra for $SO(n)$ is given by real antisymmetric $n \times n$ matrices. Show that the dimension is $\frac{1}{2}n(n-1)$. A basis for the Lie algebra is given by the matrices $L_{ij} = -L_{ji}, i, j = 1, ...n$, where $(L_{ij})_{mn} = -\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}$.

(a) Show that

$$[L_{ij}, L_{kl}] = \delta_{ik}L_{jl} - \delta_{il}L_{jk} - \delta_{jk}L_{il} + \delta_{jl}L_{ik}.$$

Dimension: The vector space of real antisymmetric $n \times n$ matrices can be thought of as being spanned by matrices with a single one in their (strict) upper triangular region, and a negative one in the transposed position in the (strict) lower triangular region - these are the matrices L_{ij}. There are $\frac{1}{2}n(n-1)$ positions in the upper triangular region, so this is the dimension of the algebra. Note we *complexify* the Lie algebra so these real dimensions are actually transformed into complex dimensions in Lie algebra theory.

Solution to (a): A calculation with index notation indeed verifies that:

$$[L_{ij}, L_{kl}] = \delta_{ik}L_{jl} - \delta_{il}L_{jk} - \delta_{jk}L_{il} + \delta_{jl}L_{ik}.$$

(b) For $n = 2r$ or $n = 2r + 1$ verify that a maximal set of matrices is given by

$$iL_{12}, \qquad iL_{34}, \qquad ..., \qquad iL_{(2r-1)2r},$$

so that the rank is r in both cases.

Solution to (b): By 'maximal', this question means 'maximally Abelian'. Using the commutation relations from (a), we can verify using an index notation argument that $[iL_{i,i+1}, iL_{j,j+1}] = 0$ for all i, j.

To show maximality, note that we can write any element of the Lie algebr as $\sum_{k,l} c_{kl} L_{kl}$.

Suppose that there exists $\sum_{k,l} c_{kl} L_{kl}$ which commutes with all $iL_{i,i+1}$. Then we have:

$$\sum_{k,l} [iL_{i,i+1}, c_{kl} L_{kl}] = i \sum_k (c_{ik} L_{i+1,k} - c_{ik} L_{i+1,k} - c_{i+1,k} L_{ik} + c_{k,i+1} L_{ik}) = 0$$

for all i. The first two terms on the right hand side immediately cancel, and using antisymmetry of the coefficients $c_{kl} = -c_{lk}$, we're left with:

$$2i \sum_k c_{k,i+1} L_{ik} = 0.$$

Since L_{ik} is a basis, we can read off $c_{k,i+1} = 0$ for all $k \neq i$. So an element can only commute with all $iL_{i,i+1}$ if it is in the span of the $iL_{i,i+1}$, and we're done.

(c) Define:

$$E_{\epsilon\eta} = L_{13} + i\epsilon L_{23} + i\eta(L_{14} + i\epsilon L_{24}), \qquad \epsilon, \eta = \pm 1,$$

and verify the commutators:

$$[iL_{12}, E_{\epsilon\eta}] = \epsilon E_{\epsilon\eta}, \qquad [iL_{34}, E_{\epsilon\eta}] = \eta E_{\epsilon\eta}, \qquad [iL_{2i-1,2i}, E_{\epsilon\eta}] = 0,$$

for $i = 3, ..., r$, so that $E_{\epsilon\eta}$ corresponds to a root vector $(\epsilon, \eta, 0, ..., 0)$.

Solution to (c): Via a short calculation using index notation. Note the result generalises to a generator defined for $m \neq n$, $m, n = 1, ..., r$:

$$E^{mn}_{\epsilon\eta} = L_{2m-1,2n-1} + i\epsilon L_{2m,2n-1} + i\eta(L_{2m-1,2n} + i\epsilon L_{2m,2n}),$$

with associated root vector $\epsilon\mathbf{e}_m + \eta\mathbf{e}_n$, with $\mathbf{e}_i$ the standard basis of $\mathbb{R}^r$.

(d) Using the notation of the previous question where $\mathbf{e}_i$ are orthogonal unit vectors in an r-dimensional space, show that $\pm\mathbf{e}_i \pm \mathbf{e}_j$ for all $i, j = 1, ..., r$, $i \neq j$ give in general $2r(r-1)$ root vectors.

Solution to (d): The generalisation in (c) shows us that indeed all $\pm\mathbf{e}_i \pm \mathbf{e}_j$ are root vectors (for $i \neq j, i, j = 1, ..., r$). There are r choices for $\mathbf{e}_i$ and $r-1$ choices for $\mathbf{e}_j$. There are 2 choices for signs ϵ and 2 choices for signs η. Finally, we note we are double-counting because i and j can be exchanged. Putting all this together gives:

$$r \cdot (r-1) \cdot 2 \cdot 2 \cdot \frac{1}{2} = 2r(r-1)$$

as required.

(e) For $n = 2r$, choose as simple roots:

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \quad ..., \quad \alpha_{r-1} = \mathbf{e}_{r-1} - \mathbf{e}_r, \quad \alpha_r = \mathbf{e}_{r-1} + \mathbf{e}_r.$$

Show that $\mathbf{e}_i - \mathbf{e}_j$, for $i < j$, and $\mathbf{e}_i + \mathbf{e}_j$ may be expressed as linear combinations of these simple roots with positive or zero integer coefficients.

(f) Show also that the other roots are given by negative linear combinations.

Solution to (e) and (f): We choose $\mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{e}_i + \mathbf{e}_j$ to be the positive roots; then all α_i are positive. We can also check that all α_i cannot be written as the sum of two positive roots via a short calculation, so that they are the simple roots for this choice of positive roots.

Note that we can write any positive root as the positive integer linear combination of simple roots since:

$$\mathbf{e}_i - \mathbf{e}_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$$

for $i < j$ and $j = 1, ..., r$, and

$$\mathbf{e}_i + \mathbf{e}_j = (\mathbf{e}_i - \mathbf{e}_r) + (\mathbf{e}_j + \mathbf{e}_r) = (\alpha_i + \alpha_{i+1} + \cdots + \alpha_j) + (\alpha_r + \alpha_{r-2} + \cdots + \alpha_j).$$

The negative roots are given by $-(\mathbf{e}_i - \mathbf{e}_j)$ and $-(\mathbf{e}_i + \mathbf{e}_j)$ so (f) follows trivially.

(g) Work out the Cartan matrix and determine the Dynkin diagram.

Solution to (g): We note that for distinct $i, j < r$:

$$(\alpha_i, \alpha_j) = (\mathbf{e}_i - \mathbf{e}_{i+1}) \cdot (\mathbf{e}_j - \mathbf{e}_{j+1}) = -\delta_{i,j+1} - \delta_{i+1,j}$$

For $i = j < r$, we have $(\alpha_i, \alpha_i) = 2$. For $i < r$, we have:

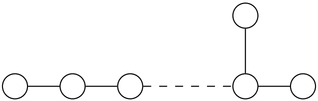
$$(\alpha_i, \alpha_r) = (\mathbf{e}_i - \mathbf{e}_{i+1}) \cdot (\mathbf{e}_{r-1} + \mathbf{e}_r) = \delta_{i,r-1} - \delta_{i+1,r-1} - \delta_{i+1,r},$$

Finally when $i = j = r$, we have $(\alpha_r, \alpha_r) = 2$.

Putting all this information together gives the $r \times r$ Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}.$$

It follows that the Dynkin diagram is:



There are a total of r nodes in the diagram. This verifies another of the Dynkin diagrams we saw earlier in the classification!

(h) For $n = 2r + 1$, verify that:

$$[iL_{12}, E_{\pm 1}] = \pm E_{\pm 1}, \qquad [iL_{2i-1,2i}, E_{\pm 1}] = 0, \qquad i = 2, ..., r,$$

where $E_{\pm 1} = L_{1,2r+1} \pm iL_{2,2r+1}$, corresponding to roots $(\pm 1, 0, ..., 0)$.

(i) Hence show that there are $2r$ additional roots in this case, $\mathbf{e}_i$, $i = 1, ..., r$.

Solution to (h) and (k): Again, through a short calculation in index notation we get the result. It is again useful to generalise to generators for $m = 1, ..., r$ given by:

$$E^m_{\pm 1} = L_{m,2m+1} \pm iL_{2,2m+1},$$

with associated roots $\pm\mathbf{e}_m$. There are $2r$ additional roots as specified in the question.

(k) In a similar fashion to the above, show that in this case we may take as simple roots:

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \quad ..., \quad \alpha_{r-1} = \mathbf{e}_{r-1} - \mathbf{e}_r, \quad \alpha_r = \mathbf{e}_r.$$

Solution to (k): We keep $\mathbf{e}_i - \mathbf{e}_j$, $\mathbf{e}_i + \mathbf{e}_j$ as positive roots before, and add $\mathbf{e}_i$ as positive roots too. Then all α_i are positive roots. It can be shown via a short calculation that none of the α_i can be written as the sum of two positive roots, hence the roots α_i are the simple roots for this choice of positive roots.

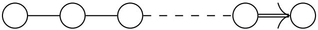
We can also verify that all positive roots can be written as positive integer combinations of the simple roots. Similarly all negative roots can be written as negative integer combinations of the simple roots.

(l) Hence obtain the Cartan matrix and determine the Dynkin diagram.

Solution to (l): Via a similar calculation to the above, the Cartan matrix is:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The Dynkin diagram is given by:



We introduce an arrow because one of the simple roots is shorter than the others (the last one is only of length 1, and the others are all of length $\sqrt{2}$). Again, we get a diagram that we saw in the classification earlier on.