Clarkson Institute Summer School:
Examples Sheet 1 - Groups: definitions, examples, and isomorphism

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Definition of a group

1. Decide which of the following sets, with the given binary operations, constitute groups. In each case, verify the group axioms, or determine which of them do not hold.

   (a) the integers $\mathbb{Z}$, with the operation $a * b = a - b$;

   (b) the complex numbers $\mathbb{C}$, with the operation $a * b = ab/(a+b)$;

   (c) the set of odd integers, with the operation $a * b = a + b + 1$.

Examples of groups from arithmetic

2. Identify the elements and construct the group table for $\mathbb{Z}_8^*$. What feature of the group table implies that the group is Abelian? Give the inverses of all the elements.

3. Find the multiplicative inverse q^{-1} of the quaternion $q = 1 + i + j + k$. Verify directly that we have $qq^{-1} = 1 = q^{-1}q$.

Examples of groups from geometry

4. Construct the group table for the set of symmetries of a non-square rectangle.

5. Construct the group table for D_4. What feature of the group table implies that the group is non-Abelian? Give the inverses of all the elements.

6. In theoretical particle physics, the Universe possesses three basic discrete symmetries: *parity inversion*, P, which sends a spatial point (x, y, z) to $(-x, -y, -z)$; *time reversal*, T, which reverses the flow of time; and *charge conjugation*, C, which changes the sign of the charge of all charged particles in the Universe. Together with the identity symmetry, they form a group of symmetries $G = \{I, C, P, T\}$. The CPT *theorem* from physics says that:
$$CPT = I.$$
   Using this theorem, construct the group table for G, and show that the group is Abelian. [*Hint: eliminate one of P, C, T in terms of the other two symmetries.*]

7. (a) Determine the orders of the chiral cubic group (the set of rotational symmetries of a cube) and the full cubic group (the set of rotational and reflectional symmetries of a cube). Are these groups Abelian or non-Abelian? Justify your answers.

   (b) Repeat part (a) for the corresponding octahedral groups. What do you notice about your answers, and why?

Examples of groups from matrix algebra

8. Show that the powers of the matrix:
$$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$
   form a group under matrix multiplication. Construct the group table.

9. Show that the following set of matrices forms a group under matrix multiplication, and construct their group table:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

10. Is it true that a set of matrices forms a group under multiplication only if all matrices in the set are invertible? Provide a proof or counterexample as appropriate.

Isomorphic groups

11. Look at the group tables you constructed in Questions 2, 4, 6, 8, and 9. In each case, identify a cyclic or dihedral group from this chapter to which the group is isomorphic.

12. Show that if $f : G_1 \to G_2$ is an isomorphism, then $f(g^{-1}) = f(g)^{-1}$ for all group elements $g \in G_1$. Deduce that C_4 and D_2 are not isomorphic groups.

13. Is Q_8 isomorphic to a cyclic or dihedral group?