Clarkson Institute Summer School:
Examples Sheet 2 - Basic properties of group elements

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Proving properties of abstract groups

1. Consider the equation $x * x = x$ for the variable x in a group. Show that the unique solution of this equation is the identity of the group.

2. Using the group axioms, prove the following laws for group elements x, y:

   (a) $(x^{-1})^{-1} = x$;

   (b) $(xy)^{-1} = y^{-1}x^{-1}$.

   How does the second property generalise to $(x_1 x_2 ... x_n)^{-1}$, where $x_1, x_2, ..., x_n$ are group elements? [*You may assume the generalised associative law here, so that $x_1 x_2 ... x_n$ is unambiguous.*]

3. Let $a_1, a_2, ...$ be elements of a group G with operation $*$. Define $S_1(a_1) = \{a_1\}$, $S_2(a_1, a_2) = \{a_1 * a_2\}$, and:

$$S_n(a_1, ..., a_n) = \left\{x * y \ \ : \ \ x \in S_p(a_1, ..., a_p),\ y \in S_q(a_{p+1}, ..., a_n),\ \ p + q = n,\ \ p, q \geq 1\right\}.$$

   (a) Explain why the set $S_n(a_1, ..., a_n)$ contains all possible elements formed from inserting brackets into the product $a_1 * a_2 * ... * a_n$. Hence, explain why the set $S_3(a_1, a_2, a_3)$ contains one element.

   (b) Using induction, prove that $S_n(a_1, ..., a_n)$ contains only a single element. Deduce the *generalised associative law*: in a group, the product $a_1 * a_2 * ... * a_n$ is independent of the placement of brackets.

Properties of group tables

4. Explain why the identity element e must appear symmetrically about the leading diagonal in a group table.

5. Using the Latin square property of groups, show that there are at most $(n-1)!^{n-1}$ groups of order n, up to isomorphism. How tight is this bound for $n = 1, 2, 3, 4$?

6. Are the following Latin squares group tables? Explain your answers.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	a	c	d	b	1
b	b	d	a	1	c
c	c	b	1	d	a
d	d	1	c	a	b

$*$	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	d	1	a	c
c	c	b	d	1	a
d	d	c	a	b	1

Power laws and periodicity

7. Find all the orders of the elements of: (a) C_4; (b) D_3; (c) Q_8; (d) the chiral tetrahedral group.

8. Let x, y be group elements. Show that: (a) x and x^{-1} have the same order; (b) x and yxy^{-1} have the same order; (c) xy and yx have the same order.

9. Show that for any finite group G, there exists an integer n, such that $x^n = e$ for all $x \in G$. The least such integer is called the *exponent* of the group. Calculate the exponents of C_6 and D_3.
10. Suppose that x, y are elements of an Abelian group, with orders n, m respectively. Is it true that the order of xy is the lowest common multiple of n, m?
11. Is it true that if all the elements of a group have finite order, then the group must be finite? Give a proof or counterexample as appropriate.

Generators and relations

12. Show that $\mathbb{Z}_6$ is generated by both $\{1\}$ and $\{2, 3\}$. What is the smallest size of a generating set for $\mathbb{Z}_8^*$?
13. Show that every group has a generating set. Is it true that every generating set of an infinite group is itself infinite? Give a proof or counterexample as appropriate.
14. A group G is generated by the distinct, non-identity elements a, b subject to the relations $a^2 = b^2 = e$ and $ab = ba$. Construct the group table, and identify the group.
15. Let r, m be the generators of D_n defined in this chapter. Show by induction that:

$$(r^\alpha m^\beta)^k = r^{\alpha S_k} m^{k\beta},$$

where $S_k = 1 + (-1)^\beta + (-1)^{2\beta} + ... + (-1)^{(k-1)\beta}$. Deduce that D_n is not a cyclic group.

16. Suppose that all the non-identity elements of a group have order two.
    (a) Show that the group must be Abelian.
    (b) Show that, if the group is finite, then it has order 2^n for some n. [*Hint: consider a minimal generating set.*] Can the group be infinite?