Clarkson Institute Summer School:
Examples Sheet 3 - Subgroup theory

Please send all comments and corrections to jmm232@cam.ac.uk.

Definition of a subgroup

1. Find all the subgroups of D_4 and describe them geometrically. Are the subgroups of order 4 all isomorphic to one another?
2. Let G be a group. Find an example of a subset $H \subseteq G$ which can be made into a group, but is not a subgroup of G.
3. Prove that the set of elements of finite order in an Abelian group is a subgroup. Must this be the case in a non-Abelian group?
4. Suppose that a group G has subgroups $H, K \leq G$. Is it true that $H \cap K$ and $H \cup K$ are subgroups? Give a proof or counterexample as appropriate.
5. Is it ever possible to express a group G in the form $G = H \cup K$ for two of its proper subgroups H, K? It is ever possible to express a group G in the form $H \cup K \cup L$ for three of its proper subgroups H, K, L?

Cosets and Lagrange's theorem

6. Find all the subgroups of the quaternion group Q_8. Construct the left cosets of one of the order four subgroups, and verify that they all have the same size, and partition the group.
7. Consider the set of matrices of the form:
$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$
where a, b, c are integers taken modulo 5.
   (a) Show that these matrices form a finite group G, and determine its order.
   (b) Show that the subset given by setting $a = c$ defines an Abelian subgroup H. Find the order of H, and the number of distinct left cosets of H in G.
   (c) Find the set of all elements of G whose square is the 3×3 identity matrix. Is the subset of G defined by $b \neq 0$ a subgroup of G?
8. Find the group table of the group generated by the matrices:
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$
Find all subgroups of the group, and give a complete set of left cosets of one of the proper, non-trivial subgroups.
9. Using Lagrange's theorem, show that:
   (a) any group of order 101 is Abelian;
   (b) if a group contains elements with orders 6 and 10, then it must have order at least 30;
   (c) the proper subgroups of any group of order 21 are all cyclic.
10. When can a finite group G contain an element of order greater than $|G|/2$?

Classification of groups

11. Show that if a non-trivial group has no proper non-trivial subgroups, then it must be cyclic and of prime order.

12. Classify all groups which have exactly three subgroups.

13. Without using Cauchy's theorem, explain why a group of even order must contain an element of order two.

14. (a) Show that if a group G has two distinct subgroups H, K, both of prime order, then $H \cap K = \{e\}$.
    (b) Now suppose that the order of the group G is $2p$, where p is a prime unequal to 2.
        (i) Explain why there must be some element $x \in G$ of order 2 generating a subgroup $H = \{e, x\}$ of order 2, and some element $y \in G$ of order p generating a subgroup $K = \{e, y, ..., y^{p-1}\}$ of order p. Explain why $H \cap K = \{e\}$.
        (ii) Explain why the left and right x-cosets of K coincide, $xK = Kx$. Deduce that $xyx^{-1} = y^k$ for some k, and hence, show that $y^{k^2} = y$. Deduce that $k = \pm 1 \pmod{p}$.
        (iii) Thus show that all groups of order $2p$ are isomorphic to either C_{2p} or D_p.

15. If every proper subgroup of a group is cyclic, must the group be cyclic?

Bonus questions

16. ($*$) Is it true that every infinite group has an infinite proper subgroup? Provide a proof or counterexample as appropriate.