Clarkson Institute Summer School: Examples Sheet 4 - Product groups

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Direct products

1. (a) Construct the group table of $C_2 \times C_3$. Find the orders of all the elements, and hence verify the isomorphism $C_2 \times C_3 \cong C_6$.

   (b) Construct the group table of $C_3 \times C_3$. Find the orders of all the elements, and hence verify that $C_3 \times C_3 \not\cong C_9$.

2. Is D_3 isomorphic to the direct product of two non-trivial groups? Exhibit two such groups, or show that none can exist.

3. Show that for any three groups G_1, G_2, G_3, we have the following:

   (a) $G_1 \times G_2 \cong G_2 \times G_1$ (hence, the direct product is 'commutative', up to isomorphism);

   (b) $G_1 \times (G_2 \times G_3) \cong (G_1 \times G_2) \times G_3$ (hence, the direct product is 'associative', up to isomorphism).

   Thus, we may be quite relaxed when manipulating direct products - e.g. it is perfectly reasonable to write $C_2 \times C_2 \times C_3 \cong C_2 \times C_3 \times C_2$ without thinking too much.

4. Which of the following direct product groups are isomorphic? Justify your answer.

$$C_{12}, \qquad C_6 \times C_2, \qquad C_3 \times C_4, \qquad C_3 \times C_2 \times C_2.$$

5. Suppose that G_1, G_2 are groups.

   (a) Let $H_1 \leq G_1, H_2 \leq G_2$ be subgroups of G_1, G_2 respectively. Show that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

   (b) Is every subgroup of $G_1 \times G_2$ of the form $H_1 \times H_2$, where $H_1 \leq G_1, H_2 \leq G_2$? Give a proof or counterexample as appropriate.

The direct product theorem

6. Let H, K be the groups of matrices, under matrix multiplication, generated by the sets:

$$\left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \qquad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

   respectively. Let G be the smallest group of matrices containing both H, K as subgroups. Show that $G \cong H \times K$.

7. Using the direct product theorem, show that $\mathbb{Z}_{35}^* \cong C_4 \times C_6$. [*Hint: consider the subgroups generated by* 8 *and* 31.]

8. Using the direct product theorem, provide an alternative proof that $C_n \times C_m \cong C_{nm}$ if and only if n, m share no common factors except one.

9. Show that the full cubic group is isomorphic to $C_2 \times O$, where O is the chiral cubic group. [*Hint: $-I$, where I is the identity, is a symmetry of the cube.*]

Classification of finite Abelian groups

10. List all Abelian groups of order 108.

11. A finite Abelian group has order 16, and contains an element of order 8, but no elements of order 16. What is the group?

12. Let G be a finite Abelian group whose order is a power of two. Show that G has exactly one element of order two if and only if G is cyclic.

Classification of groups

13. (a) A group G is called *decomposable* if it isomorphic to an external direct product:

$$G \cong H \times K,$$

where H, K are non-trivial groups; a group is called *indecomposable* otherwise. Prove that every finite group can be written in the form:

$$G \cong G_1 \times G_2 \times ... \times G_n,$$

where G_i is an indecomposable group. [*Well beyond the scope of this course, the Krull-Schmidt theorem states that these indecomposable factors G_i are unique up to isomorphism and reordering.*]

(b) Show that that Q_8 is indecomposable.

(c) Let $r^\alpha m^\beta, r^{\alpha'} m^{\beta'} \in D_n$. Show that:

$$(r^\alpha m^\beta)(r^{\alpha'} m^{\beta'}) = (r^{\alpha'} m^{\beta'})(r^\alpha m^\beta)$$

if and only if $\alpha + (-1)^\beta \alpha' = \alpha' + (-1)^{\beta'} \alpha$ modulo n. Deduce that:

- rotations always commute;
- a rotation r^α and a reflection $r^{\alpha'} m$ commute if and only if $2\alpha = 0$ modulo n;
- two reflections $r^\alpha m$ and $r^{\alpha'} m$ commute if and only if $2(\alpha - \alpha') = 0$ modulo n.

Hence, argue that D_n is decomposable if n is twice an odd number, else it is indecomposable.

(d) Using parts (b) and (c), make a complete list of indecomposable groups up to and including order 11.

14. Is it true that if a group's non-identity elements all have order three, then the group must be Abelian? Provide a proof or counterexample as appropriate.