Clarkson Institute Summer School:
Examples Sheet 5 - Conjugacy and normality

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Conjugate elements and normal subgroups

1. Starting from first principles, partition the elements of D_6 into conjugacy classes, explaining your reasoning both algebraically and geometrically. Hence, identify the normal subgroups of D_6.

2. Partition the elements of Q_8 into conjugacy classes. Hence, identify the normal subgroups of Q_8.

3. Show that if $H \leq G$ is a subgroup of a group G which only has two distinct left cosets in G, then H must be normal.

4. The *centre* $Z(G)$ of a group G is the set of all elements which lie in singleton conjugacy classes,
$$Z(G) = \{z \in G : gzg^{-1} = z \text{ for all } g \in G\}.$$
   (a) Show that $Z(G)$ is an Abelian normal subgroup of G.
   (b) Show that if H is a normal subgroup of G with $|H| = 2$, then $H \trianglelefteq Z(G)$.
   (c) Determine $Z(D_4)$.

5. Which finite groups have exactly two conjugacy classes?

Quotient groups

6. Using your results from Questions 1 and 2, construct all possible quotients of D_6 and Q_8. Identify standard cyclic and dihedral groups to which the quotients are isomorphic.

7. Let m be a reflection in D_4. Show that $\{e, m\}$ is not a normal subgroup. Construct its left cosets, and hence show that coset multiplication is not well-defined in this instance.

8. In theoretical physics, a *Lorentz boost* is a symmetry transformation of $(1+1)$-dimensional spacetime represented by the matrix:
$$\begin{pmatrix} \cosh(\phi) & -\sinh(\phi) \\ -\sinh(\phi) & \cosh(\phi) \end{pmatrix},$$
where ϕ is a real parameter, called the *rapidity* of the Lorentz boost. It represents a transformation between a stationary frame of reference, with coordinates (x, ct), and a moving frame of reference, (x', ct').
   (a) Show that the set H of Lorentz boosts forms a group.
   (b) The set of Lorentz boosts is now extended with time-reversal symmetry T, and parity inversion P, represented by the matrices:
$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
   Let G be the group generated by the elements of H, together with T and P. Show that $H \trianglelefteq G$ is a normal subgroup of G. Is the group K, generated by T and P only, a normal subgroup of G?
   (c) Construct the quotient group G/H, and identify it with a standard group.

9. By naturally identifying C_n with a subgroup of C_{nm}, prove that:

$$\frac{C_{nm}}{C_n} \cong C_m.$$

   Deduce the structure of all possible quotients of C_{24}.

10. Show that both H and K may be naturally identified with normal subgroups of $H \times K$. Hence, prove that:

$$\frac{H \times K}{H} \cong K \qquad \text{and} \qquad \frac{H \times K}{K} \cong H.$$

11. Let G be a group. Show that $G/Z(G)$ is cyclic if and only if G is Abelian.

Common misconceptions about quotients

12. Is it true that for any group G with a normal subgroup $H \trianglelefteq G$, we must have:

$$\frac{G}{H} \times H \cong G?$$

   Give a proof or counterexample as appropriate.

13. If $K \trianglelefteq H$, and $H \trianglelefteq G$, is is true that $K \trianglelefteq G$? Provide a proof or counterexample as appropriate.

14. Let $H, K \trianglelefteq G$ be normal subgroups of a group G with $H \cong K$. Is it true that $G/H \cong G/K$? Provide a proof or counterexample as appropriate.

Bonus questions

15. ($*$) Let G be a finite non-Abelian group.
    (a) Explain why the size of $Z(G)$ can be at most a quarter the size of G.
    (b) Hence, show that the chance of picking two elements from G at random that commute is $5/8$. Show that this bound cannot be improved.